Math 108 - Geometric Combinatorics Nadia Lafrenière

The game of Hex, a ham & cheese sandwich, and the stolen necklace

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We apply Brouwer's fixed point and Borsuk-Ulam Theorem to fair-division and game theory problems.

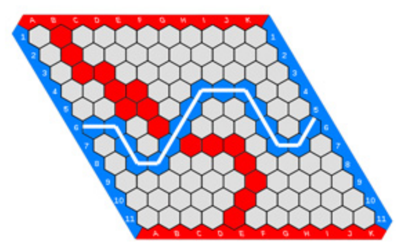
The game of Hex

Hex is a board game played on a hexagonal grid, in which players take turn placing tokens on the board (anywhere), with the goal of making a path from one end of the board to the other.

Observation: At most one player wins (otherwise, paths cross and overlap).

Theorem (Nash, 1949; proof here by Gale, 1979)

Hex cannot end in a draw.



Source picture: Jean-Luc W on Wikipedia

Proof

Assume that there is no winner, and that we work on the grid on the right. Since there is no winner and the game ends, every tile of the board is either red or blue.

We partition the tiles in 4 sets:

- BL : blue tiles connected by a path to the left side of the board.
- BR: other blue tiles
- RB: red tiles connected by a path to the top of the board.
- RT: other red tiles.

We also define the vectors:

→ e_1 : one step to the right

↖ e_2: one step above, where "above" is considered as parallel to the side.

Then, we have the function below, defined for points that are the barycenter of the tiles:

$$f(v) = \begin{cases} v + e_1 & \text{if } v \in BL \\ v - e_1 & \text{if } v \in BR \\ v + e_2 & \text{if } v \in RB \\ v - e_2 & \text{if } v \in RT. \end{cases}$$

We extend f affinely, as to create a continuous function on the "disk" the board.
↳ using the closest tiles (up to 3).

A fixed point can only occur when on an edge between two tiles, such that either one is in BL and one in BR, or one is in RB and the other one in RT.

However, by definition of the sets, tiles in BL and BR cannot be adjacent (nor can the tiles in RB and in RT). Hence, there is no fixed point.

But f is a continuous function on (something homeomorphic to) the disk and, by Brouwer's fixed point theorem, there is a fixed point.

Hence, there must be either a blue or a red path across the board, and the game does not end in a draw.

The Ham Sandwich Theorem

We discuss an n-dimensional version of a theorem that is refered to as the pancake theorem in $\mathbb{R}^2$ or as the ham sandwich theorem. It says that there are always a hyperplane that cuts a 3-dimensional sandwich such that each of the three ingredients [bread, ham, lettuce?-cheese?-butter?] is equidistributed between the two parts.

Theorem (Stone-Tukey, 1942; Ham sandwich Theorem)

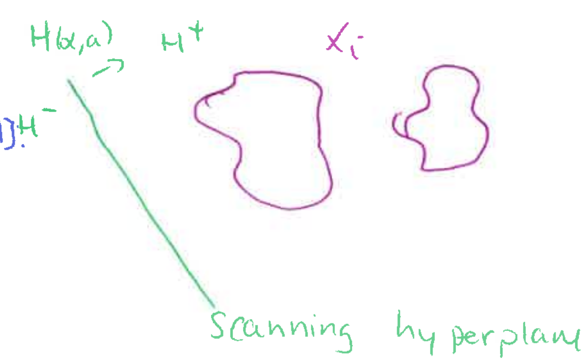
Let $X_1, X_2, \dots, X_n$ be bounded and measurable parts in $\mathbb{R}^n$. Then, there exists at least one hyperplane that divides each of $X_1, \dots, X_n$ exactly in half.

Proof

An affine hyperplane is defined by a fixed vector α (we can assume α is a unit vector), an a translation factor a in $\mathbb{R}$.

Then, $H(\alpha, a) = \{v \in \mathbb{R}^n : \alpha \cdot v = a\}$, and $H^+(\alpha, a) = \{v \in \mathbb{R}^n : \alpha \cdot v > a\}$.

For fixed α and i, consider the function of a that gives the measure of $H^+(\alpha, a) \cap X_i$. This function is decreasing over $\mathbb{R}$ and continuous. By the intermediate value theorem, it takes all the values in $[0, |X_i|]$.



Moreover,

$$H^+(-\alpha, -a) = \{v \in \mathbb{R}^n : -\alpha \cdot v > -a\}$$
$$= \{v \in \mathbb{R}^n : \alpha \cdot v < a\} = H^-(\alpha, a)$$

Given a unit vector α in $\mathbb{R}^n$ (so $\alpha \in S^{n-1}$), define $g(\alpha)$ to be the value a for which $|H^+(\alpha, a) \cap X_1| = |X_1|/2$.

Then, $f_i(\alpha) = |H^+(\alpha, g(\alpha)) \cap X_i|$ is a continuous function for each i, and

$$f : S^{n-1} \to \mathbb{R}^{n-1}$$
$$\alpha \mapsto (f_2(\alpha), \dots, f_n(\alpha))$$
$$= (|H^+(\alpha, g(\alpha)) \cap X_2|, \dots, |H^+(\alpha, g(\alpha)) \cap X_n|)$$

is continuous.

From the Borsuk-Ulam theorem, there exists $\alpha \in S^{n-1}$ for which

$f(\alpha) = f(-\alpha)$; i.e. for which $f_i(\alpha) = |H^+(\alpha, g(\alpha)) \cap X_i| = |H^-(\alpha, g(\alpha)) \cap X_i| = \frac{1}{2}|X_i|$

Because this is true for all $i \in \{1, 2, \dots, n\}$, there exists a unit vector α defining a hyperplane $H(\alpha, g(\alpha))$ splitting each of $X_1, \dots, X_n$ into two equal parts.

The stolen necklace

We present a discrete analogue of the ham sandwich theorem, and then apply it to a combinatorial problem.

Theorem (Discrete Ham Sandwich)

Let $X_1, \dots, X_n$ be finite sets of points in general position in $\mathbb{R}^n$*. Then, there exists a hyperplane that divides each of $X_1, \dots, X_n$ into two sets of the same size, i.e. each half-space contains exactly $\lfloor \frac{|X_i|}{2} \rfloor$ points of X_i, for each $i \in \{1, 2, \dots, n\}$.

* so that, on each hyperplane, there are at most n points.

Problem: Two thieves stole a pricy necklace and would like to fairly split its value. There are n different types of beads, and the number of beads of each type is even. Because they don't know the value of the beads they will split the set of beads of each type in half.

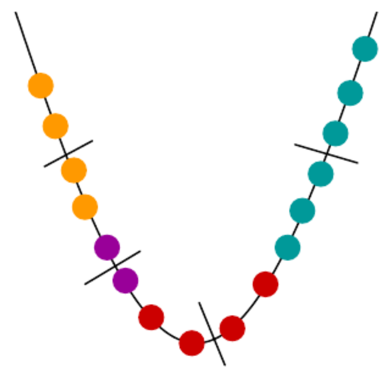
The chain is made of gold, so they want to cut it as few times as possible. The necklace has an opening mechanism.

How many cuts are needed?

Corollary (of discrete ham sandwich theorem; Goldberg & West 1985)

n cuts are always sufficient to split evenly the n types of beads of the necklace

n cuts are sometimes needed as well, as in the example to the right.



Proof

Consider the curve:

$$\gamma: \mathbb{R} \longrightarrow \mathbb{R}^n$$
$$t \longmapsto (t, t^2, \dots, t^n).$$

Let H be a hyperplane in $\mathbb{R}^n$. Then, H intersects γ in at most n points; $H = \{v \in \mathbb{R}^n \mid \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = a\}$ intersects with γ exactly when $\alpha_1 t + \dots + \alpha_n t^n = a$. Since this is a polynomial of degree n, there are at most n solutions.

Hence, placing the open necklace along γ, there is a hyperplane that splits the necklace in two parts with half the beads of each type in each part, and that requires cutting the necklace at most n times. ∎

References: [Matousek] §3

[Gal79]

[GW85]