Math 108 - Geometric Combinatorics Nadia Lafrenière

The Borsuk-Ulam Theorem & Tucker's Lemma 2/17/2023

The use of the Borsuk-Ulam theorem to solve some fair division and graph theory problems is seen as the advent of topological combinatorics

The Borsuk-Ulam Theorem

Corollary (of theorem below; or "how I remember its content")
At any time, there are two diametrically opposed points on planet Earth where the pressure and the temperature at these points are the same.
Those points change in time.

Which statement does it follow from?

Corollary (of theorem below).
Take an inflated ball, deflate it, and put it on a table. There are two diametrically opposed points of the ball that lie on top of eachother.

Picture credit: [Mat03]

We should state the theorem. We won't prove it because it would require hard topological tools, but we prove that the multiple statements are equivalent.

Theorem (Borsuk 1933; after a conjecture of Ulam, 1933)

An antipodal mapping f is a continuous function such that $f(x) = -f(-x)$, for all x. Then,

(1) There is no antipodal mapping $f: S^n \to S^{n-1}$

Sphere of dimension n; lives in $\mathbb{R}^{n+1}$.

(2) For each antipodal mapping $f: S^n \to \mathbb{R}^n$, there exists $x \in S^n$ s.t. $f(x) = (0, \dots, 0)$

(3) For any continuous mapping $f: S^n \to \mathbb{R}^n$, there exists x such that $f(x) = f(-x)$.

(4) (Also Schnirelman–Ljusternik Theorem) Any open (resp. closed) cover of S^n by $n+1$ sets is such that at least one set contains a pair of antipodal points.

Proof (of equivalence)

$$\begin{array}{ccc} (1) & \Rightarrow & (2) \\ \Uparrow & & \Downarrow \\ (4) & \Leftarrow & (3) \end{array}$$

$f: S^n \to \mathbb{R}^n$

(1)⇒(2) By contrapositive let f be an antipodal mapping that never vanishes. Then,

$$g(x) = \frac{f(x)}{\|f(x)\|}$$

is a well-defined antipodal mapping $S^n \to S^{n-1}$

(2)⇒(3) By contrapositive let f be a continuous mapping for which there are no x s.t. $f(x) = f(-x)$. Define

$$g(x) = f(x) - f(-x).$$

Then, g is an antipodal mapping on the same domain and image that never vanishes, contradicting (2).

(3) ⇒ (4) We prove it by contradiction, and for closed covers. The case of open covers is similar, using that $S^n \setminus A_i$ is open when A_i is closed.

Assume S^n has a cover $A = \{A_1, A_2, \dots, A_{n+1}\}$ in which no closed set A_i contains a pair of antipodal points.

Let $x \in A_i$. Then, $-x \notin A_i$, and $d(x, A_i) > 0$.

Define $f: S^n \to \mathbb{R}^n$

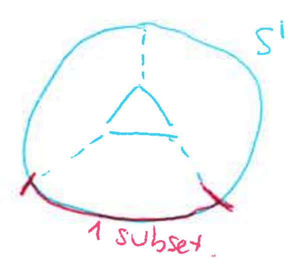
$$f(x) = (d(x, A_1), \dots, d(x, A_n))$$

Using (3), there exists x such that $f(x) = f(-x)$.

Since A_i cannot contain both x and $-x$, $d(x, A_i) \neq 0$ for all $i \in \{1, \dots, n\}$. Hence, x and $-x$ belong to A_{n+1}, showing that at least one subset contains a pair of antipodal points.

Remark: There exists a cover of S^n by $n+2$ closed subsets such that no pair of antipodal points belong to the same subset.

Idea: put a simplex in the middle of the sphere, and expand it until the edges touch the sphere. This creates a cover of the sphere into $n+2$ subsets, the facets.



(4) ⇒ (1). By contrapositive.

Let f be an antipodal mapping $S^n \to S^{n-1}$.

Let $\{A_1, \dots, A_{n+1}\}$ be a cover by closed sets of S^{n-1}, containing no antipodal points (it exists, by the remark above).

Then, $f^{-1}(A_1), \dots, f^{-1}(A_{n+1})$ is a collection of closed sets that cover S^n. Also, for any i, $f^{-1}(A_i)$ does not contain both x and $-x$ (for some x): otherwise, $-f(-x) = f(x) \in A_i$ and $f(-x) \in A_i$, so $f(-x)$ and its antipode belong to A_i, contradicting the hypothesis. ◻

A discrete analog: Tucker's Lemma

Triangulations, again!

Theorem (Tucker's lemma, 1946)

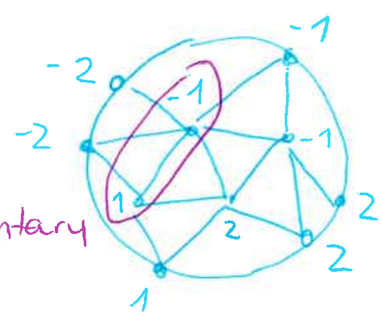
Let T be a triangulation (decomposition into simplices) of B^n that is antipodally symmetric on the boundary. Let

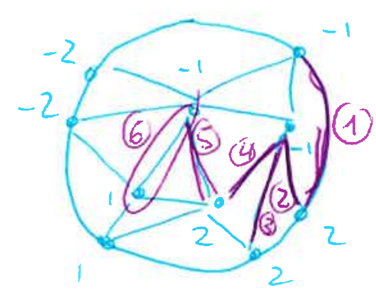
$$\lambda : V(T) \longrightarrow \{\pm 1, \pm 2, \dots, \pm n\}$$

be a labeling of the vertices of T with $\lambda(v) = -\lambda(-v)$ for all $v \in \partial B^n$. Then, there exists an edge in T that is complementary, i.e. its two endpoints v_1 and v_2 satisfy $\lambda(v_1) + \lambda(v_2) = 0$.

Example: $n = 2$

complementary edge





Sketch of proof, for $n=2$ (generalizes to higher dimension)

Goal: find the edge, by drawing a path towards it.

- Start on the boundary: if there is a complimentary edge, we are done. Otherwise, it means that we use all of $+1, -1, +2$ and -2. The edges $\{+1, -2\}$ and $\{-1, +2\}$ must appear (because of antipodality and the absence of $\{+1,-1\}$ and $\{+2, -2\}$). We can even show that $\{+1, -2\}$ appears an odd number of times.
- Start at a $\{-1, +2\}$ edge. If the third vertex of the simplex is
  - 1 or -2: we have found the complimentary edge.
  - -1 or 2: there is a second edge labeled $\{-1, +2\}$ in that simplex. Repeat the process with the other simplex to which e belongs.

5

- We need to show that this process terminates by finding a complimentary edge. For this it is enough to show that
  - in this process, no edge is used twice. (not obvious)
  - if we "fall out" of the ball, we can start over with an unused edge on the boundary. This is possible because there is an odd number of $\{-1, +2\}$ edges on the boundary.

Therefore, the algorithm always terminates by finding a complimentary edge. ∎

Theorem

The Borsuk-Ulam Theorem is equivalent to Tucker's lemma.

Proof: Homework.

References: [Lon13] M. de Longueville. A Course in Topological Combinatorics, 2013.

[Mat03] J. Matoušek. Using the Borsuk-Ulam Theorem, 2003.