Math 108 - Geometric Combinatorics

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Creating tilings

**Goal:** Find non-convex tiles that tile the plane.

Tiling by translations

Tiles that tile the plane by translation:

Examples: □ , ⬭ , ⫸ , ⬡ , ✚ , 🧩

(badly drawn puzzle piece).

Remark: No triangle tiles the plane by translation :

gap.

Theorem (Beauquier & Nivat, 1991)

A tile tiles the plane using only translations if and only if it

is a "parallelogram" or a "hexagon" in the following sense: there

exists six consecutive points points a, b, c, d, e, f (with potentially both $b=c$ and $e=f$) on its

boundary with the segments ab and ed (as well as bc and fe, and

af and cd) have the same length and same shape (orientation).

Examples
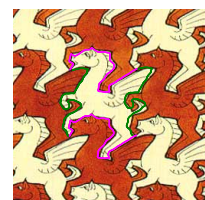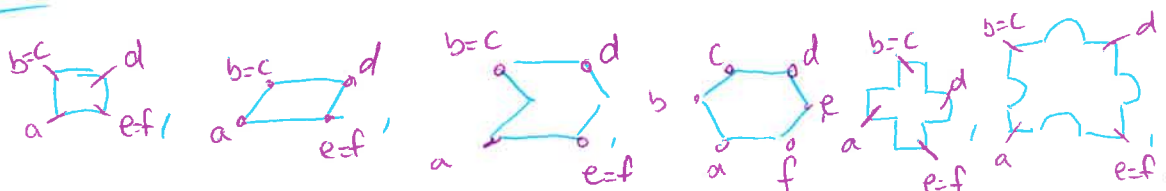
Tilings by rotations

Unlike with rotations, there is no known necessary and sufficient criterion for a tile to tile the plane using translations and rotations.

Theorem (Conway Criterion, ~1975)

A tile allows for a plane tiling using translations and reflections if there exists six consecutive points on the boundary a, b, c, d, e, f (at least three of which are distinct) with

- ab and ed have the same length and shape (including orientation).
- bc, cd, ef and af are all invariant under rotation by $180°$.
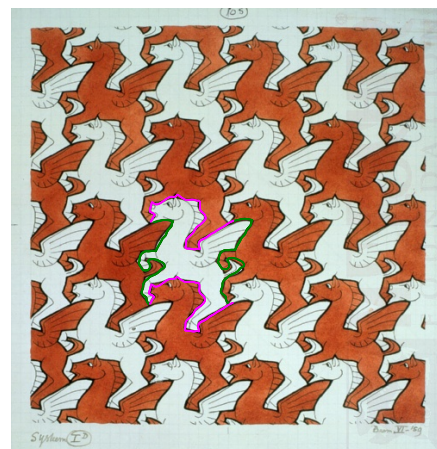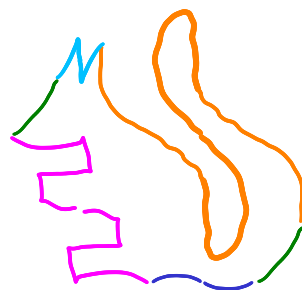
Examples
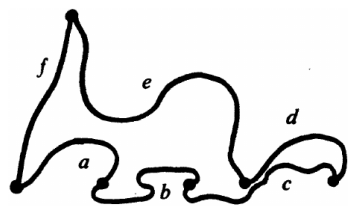
Credit: D. Schattscheiner, Will it tile? Try the Conway Criterion.
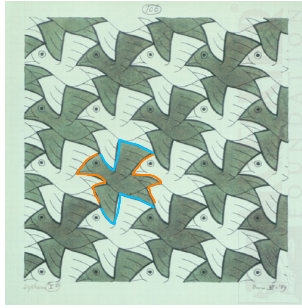

Credit: M.C. Escher

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This is certainly not a necessary condition, as some tiles meet the Beauquier-Nivat theorem and do not meet the Conway criterion.

Corollary

- All triangles tile the plane using translations and rotations.

- All quadrilaterals tile the plane using translations and rotations.

- Hexagons with one pair of opposites sides that are parallel and congruent tile the plane using translations and rotations.
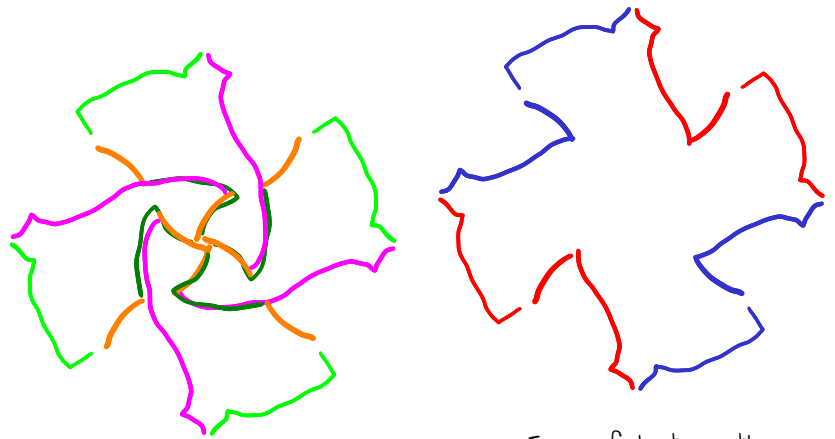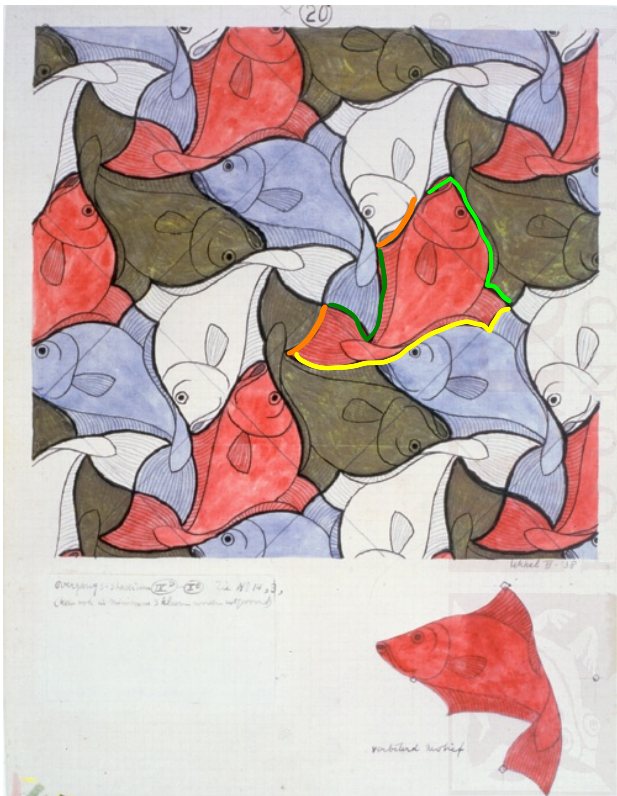
Credit: M.C. Escher

Artistic drawings

Many artistic drawings use simple tiles, that are then modified to make it looks good.

Example: A pentagonal fish, by M.C. Escher.

Copyright: M.C. Escher

Four fish together tile the plane by translation
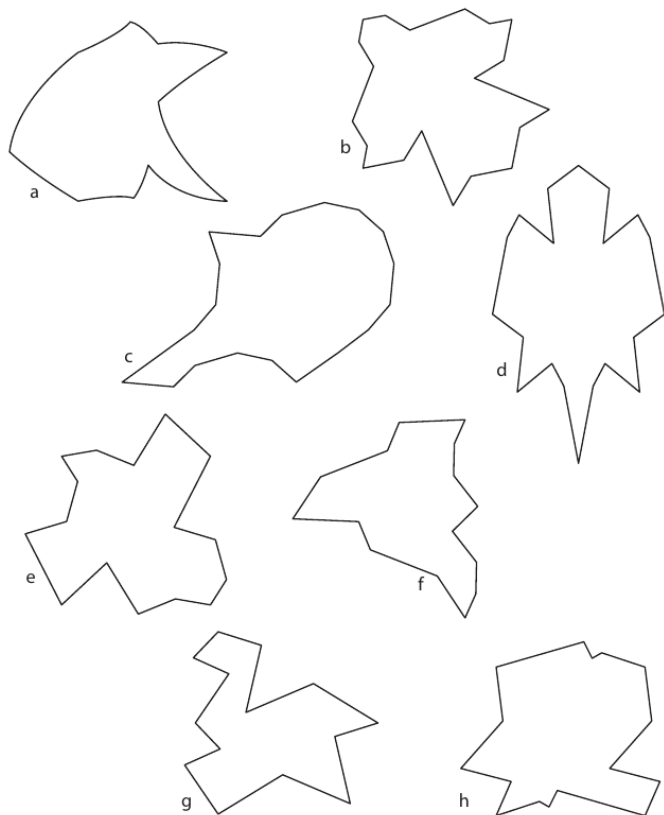
90° angles

orange sides are congruent and parallel

Steps to make a tiling by hand, or using computer drawing tools:

- Choose a tile that tiles the plane, or build one following the Conway criterion or the Beauquier-Nivat theorem.
- Deform it into something that resembles what you want to draw, making sure it keeps meeting the criteria for tiling.
- Draw the interior of the shape + color it. It is common to use different colors for different copies of the shape.

Worksheet 11.1. Finding motifs for a tile shape

For each tile shape below, think of a real-world motif that could be used for it, and draw in interior details. It may help to rotate the sheet to look at the shapes in different orientations and to consider different viewpoints (e.g., top view vs. side view).

Source of activity:
Fathauer, Tessellations.

Reference: R. Fathauer, Tessellations: Mathematics, Art and Recreation. Pictures of M.C. Escher.