Math 108 - Geometric Combinatorics Nadia Lafrenière

Tiling with convex polygons 2/13/2023

We start discussing what tiles allow to tile an arbitrary large area of the plane, without overlap nor holes.
Translations, rotations and symmetries are allowed.

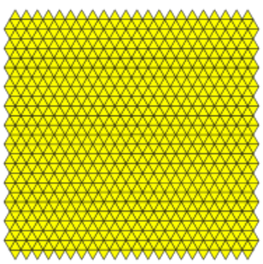
Today: convex polygons.

We assume that all tilings are edge-to-edge, unless specified otherwise, meaning that adjacent tiles only share full sides.

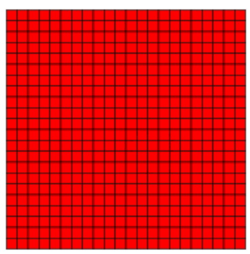
Regular polygons

not edge-to-edge

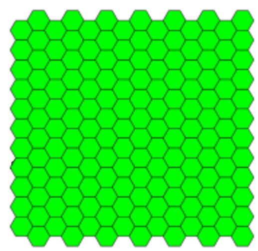
What regular polygons tile the plane?



triangular lattice



square lattice



hexagonal lattice

pictures from wikipedia

Proposition

The only regular polygons that tile the plane are the triangle, the square and the hexagon.

Proof

The angles of a regular n-gon all have $\frac{n-2}{n} \cdot 180°$.

Therefore, at the intersections, there are $\frac{360}{\frac{n-2}{n}180} = \frac{2n}{n-2}$ polygons.

The quantity $\frac{2n}{n-2}$, with $n \geq 3$, is an integer only for $n = 3, 4, 6$.

pictures from Quanta Magazine

Non-regular convex polygons

Goal: List all convex polygons that tile the plane!

Number of sides

Can convex polygons with an arbitrary number of sides tile the plane?

Your hypothesis:

Proposition

If a convex polygon tiles the plane, it has 3, 4, 5 or 6 sides.

Proof

We count the number of tiles that meet at vertices.

For an edge-to-edge tiling to exist, we need that at least 3 tiles meet at each vertex.

The average number of tiles that meet at one vertex is

full rotation

$$\frac{360^\circ}{\frac{n-2}{n}\cdot 180^\circ} = \frac{2n}{n-2}$$

average angle

If $n \geq 7$, this is < 3, and an edge-to-edge tiling is not possible.

Triangles

Theorem

All triangles tile the plane.

Lemma

All parallelogram tile the plane by translation.

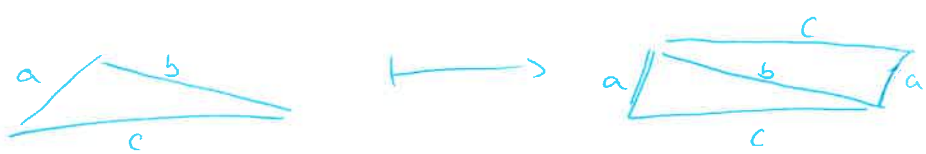
Sketch of proof

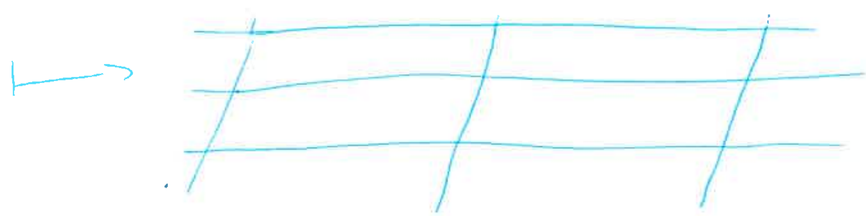
A parallelogram is a deformation of a square.

Apply the same deformation to the square grid.

Proof of Theorem

Main idea: build a parallelogram, tile the plane with it.





Quadrilaterals

What quadrilaterals tile the plane:

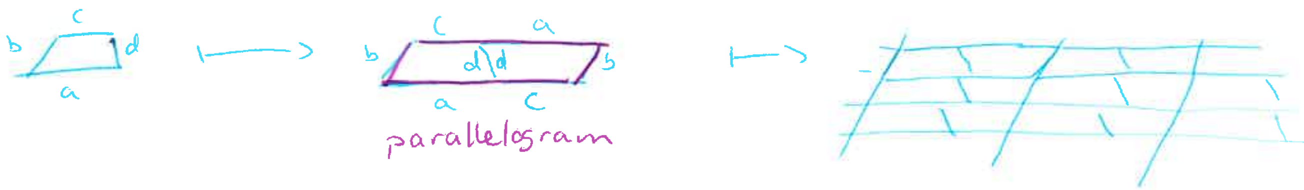
- Squares
- Rectangles?
- Parallelograms?
- Trapezoids?
- Any convex quadrilateral?
- Any quadrilateral?

Theorem

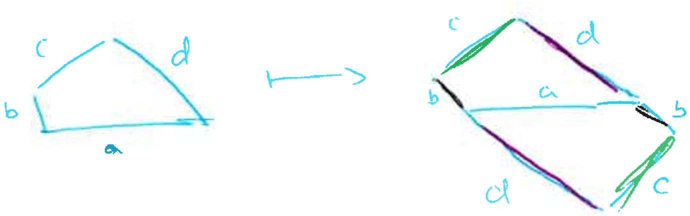
Any quadrilateral tile the plane!

Proof

- Parallelograms tile the plane, because they are deformations of the square
- If there is a pair of parallel sides, one can build an infinite strip with one rotation, so it tiles the plane.



- If there is no parallel sides, we also rotate along any side:



hexagon with 3 pairs of parallel sides = deformation of

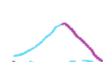
- grid

- Even non-convex shapes tile the plane

Notice that the corresponding sides are parallel.

We show that the 'a' sides are parallel, which happens if and only if $\alpha+\beta+\gamma+\delta=180^\circ$

We know: $\delta+\sigma+\alpha+\mu=360^\circ$ (interior angles) and $\beta+\tau+\gamma=180^\circ$



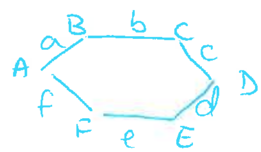
Also, $\mu+\tau+\sigma=360^\circ$, because

Hence, $\alpha+\beta+\gamma+\delta = (\delta+\sigma+\alpha+\mu)+(\beta+\tau+\gamma)-(\mu+\tau+\sigma) = 360^\circ+180^\circ-360^\circ = 180^\circ$

Therefore, gluing the two concave shapes, we get a concave hexagon with three pairs of parallel sides. This allows for tiling the plane with only translations (I'm not proving this claim).

Hexagons

Theorem



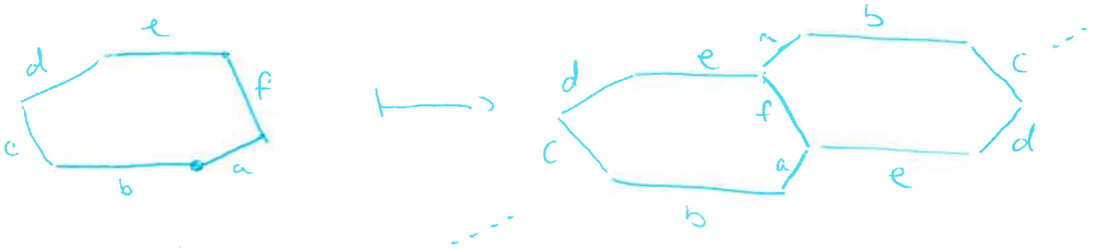
Small letters: edges (sides)
Capital letters: vertices (angles)

A convex hexagon tiles the plane (edge-to-edge) iff either

- a pair of opposite sides are parallel and have the same measure
- there are three pairs of adjacent sides with the same measure. The angle in the middle of the pair is 120°.
- a pair of opposites sides have the same measure (e.g. b and e), two other sides at distance 2 have the same measure (e.g. d and f) and the sum of angles B, C and F (or A, D and E) is 360°.

Sketch of proof

- To be able to tile with hexagons, we need at least 2 sides with the same measure (mostly because the operations rely on gluing sides of same length together)
- If the sides with the same measure are opposite and parallel, then form an infinite strip like with tra



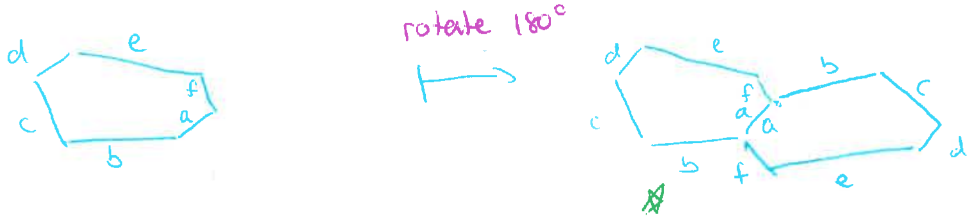
rotation of 180° + translation

$b=e$, $b \parallel e$

- If the sides are not parallel: $b=e$, $b \nparallel e$

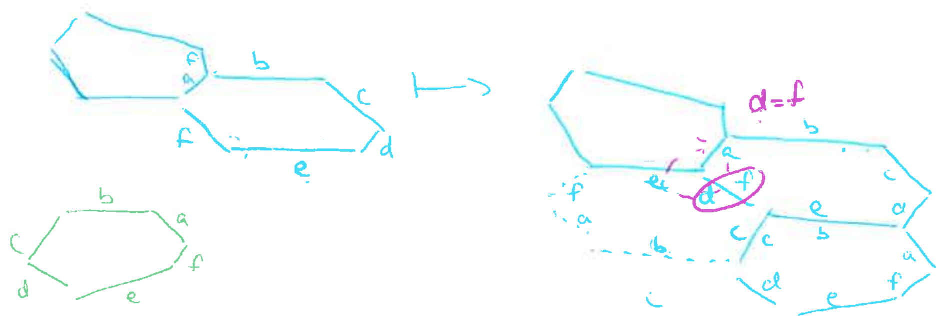
rotate 180° + transvection.

rotate 180°



Cannot rotate, nor reflect both tiles to create something in the area with ✩

reflect one tile + glue b on e.

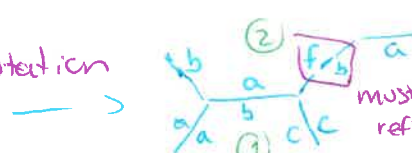


=> A+B+E must be 360° to tile properly

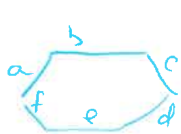
- If the sides with same measure are not opposite: rotate 60°.

$a=b$

rotation



must reflect, unless $a=b=c=d=e=f$

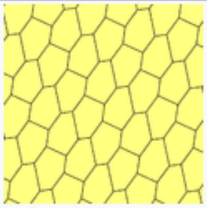
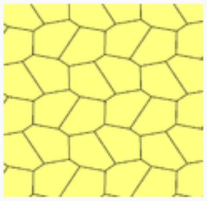
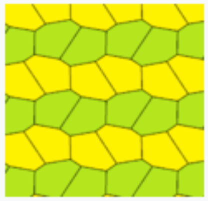
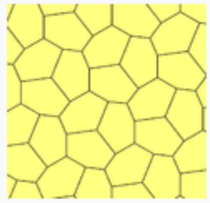
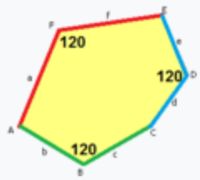


$3 \cdot \text{angle}(AB) = 360°$

$\Rightarrow \text{angle}(ab) = \text{angle}(cd) = \text{angle}(ef) = 120°$

Summary

3 types of monohedral convex hexagonal tilings

1	2		3
p2, 2222	pgg, 22×	p2, 2222	p3, 333
			
 b = e B + C + D = 360°	 b = e, d = f B + C + E = 360°		 a = f, b = c, d = e B = D = F = 120°
 2-tile lattice	 4-tile lattice		 3-tile lattice

credit : Wikipedia

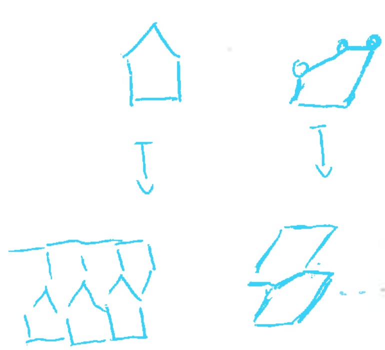
Pentagons

What pentagons tile the plane?

- Open problem 1918 - 2017(?)
- Several claims that "we had found all pentagons that tile the plane", followed by new pentagons

One example:

"Type 1": Two parallel sides



History:

- Reinhardt, 1918: Type 1 to 5 (only 2 types are edge-to-edge)
  people did not search for more types immediately

- Kershner, 1968: Type 6 to 8 (all edge-to-edge)
  Claim that list was complete, but did not write the proof because it would be too long

- Kershner's claim is popularized in Scientific American (1975)

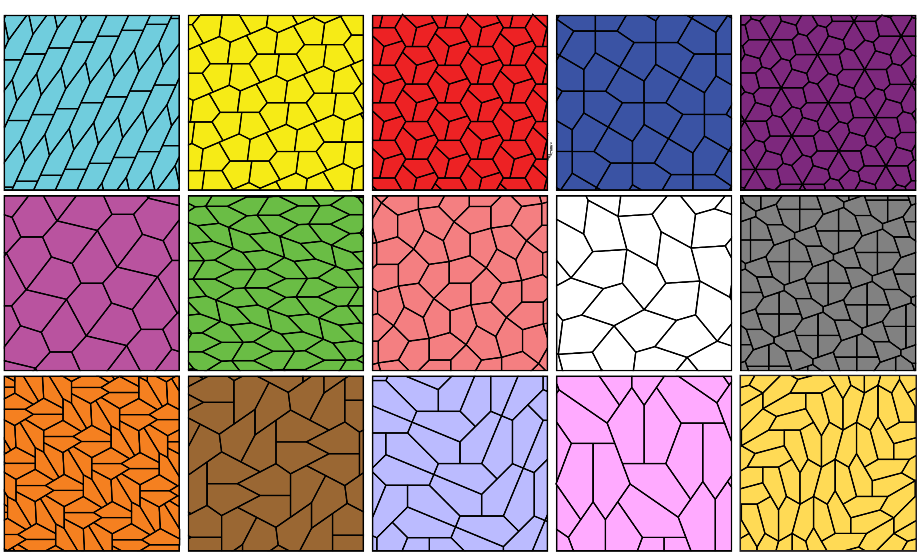
- James, 1975 (an engineer): Type 10, not edge to edge.

- Rice, 1977 (a house wife with no math background):
  Type 9 (edge-to-edge), as well as 11, 12, 13

- Stein, 1985 (not edge to edge)

- Mann, McLoud, Von Derau, 2015: type 15
  An undergrad!

- Rao, 2017, through computer exhaustion of all 371 ways vertices can meet: claim that no other pentagon can tile the plane, still under peer review.



picture from wikipedia

Pictures: Wikipedia, Quanta Magazine