Math 108 - Geometric Combinatorics — Nadia Lafrenière

Matroid invariants — 2/6/2023, 2/8/2023

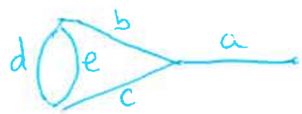
Definition

Let $M=(E,\mathcal{I})$ be a matroid.

- A **basis** is a maximal independent set.
- A **circuit** is a minimal dependent set.
- The **rank** of $A \subseteq E$ is $\max\{|X| : X \subseteq A, X \in \mathcal{I}\}$.
- The rank of M is the rank of E.

Recall that all bases have the same rank (by the third property of matroids)

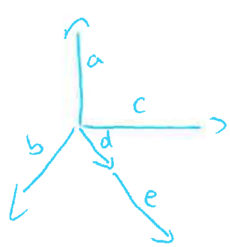
Example

Consider the graph 

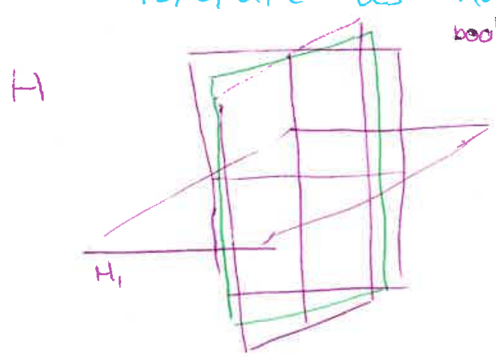
The bases are the spanning trees: abc, abd, abe, acd, ace

The circuits are the minimal cycles: bcd, bce, de.

As seen in the last lecture, this matroid is isomorphic to the linear matroid $\{a=(1,0,0), b=(0,1,0), c=(0,0,1), d=(0,\frac{1}{2},\frac{1}{2}), e=(0,1,1)\}$

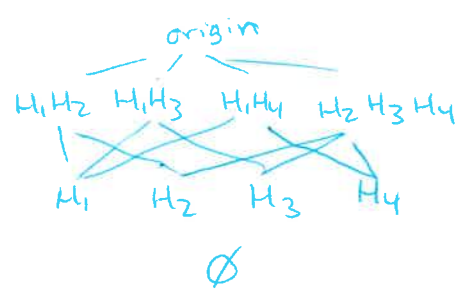


One can build a central hyperplane arrangement with a,b,c,d,e as normal vectors.



boolean arrangement + 1 hyperplane H_4

intersection poset



Definition

Fix a total order $\trianglelefteq$ on the ground set E.

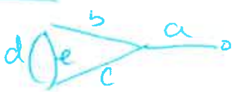
A broken circuit is a subset of E of the form

$$\hat{c} = c \setminus \{\min_{\trianglelefteq} c\},$$

where c is a circuit.

A broken circuit is an independent set.

Example

$G =$ 

, and fix $a \trianglelefteq b \trianglelefteq c \trianglelefteq d \trianglelefteq e$.

circuits	broken circuits
bcd	cd
bce	ce
de	e

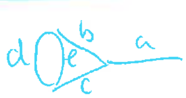
Recall that a simplicial complex Δ is a set of simplices (line segments, triangles, tetrahedra, ...) such that

- Every face of a simplex from Δ is also in Δ
- The non-empty intersection of any two simplices of Δ is a face of Δ that belong to both simplices.

The broken circuit complex is the simplicial complex

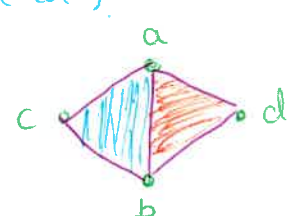
$$BC(M) = \{ A \subseteq E : A \text{ contains no broken circuit} \}.$$

Example



$BC(M) = \{ A \subseteq E : A \text{ avoids } e \text{ and } cd \}$

Maximal simplices: abc, abd,



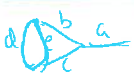
Dimension	# Faces
−1 (empty set)	1
0 (vertices)	4
1 (edges)	5
2 (facets)	2
3+	0

extended f-vector (f-vector with the # of faces of dim. 0)

$(1, 4, 5, 2)$

The independence complex is the simplicial complex of independent sets. Its simplices of highest dimension are the bases.

Example



Dimension	# faces	list
2	5	abc, abd, acd, abe, ace
1	9	ab, ac, ad, ae, bc, bd, be, cd, ce
0	5	a, b, c, d, e
−1	1	$\emptyset$

extended f-vector: $(1,5,9,5)$

The characteristic polynomial of a matroid $M=(E,\mathcal{I})$ is given by

$$\chi_M(q) = \sum_{A \subseteq E} (-1)^{|A|} q^{\operatorname{rank}(M)-\operatorname{rank}(A)}.$$

Example

Subsets of E sorted by rank

rank 0	rank 1	rank 2	rank 3	
$\boxed{\emptyset}$	$\boxed{a}$, $\boxed{b}$, $\boxed{c}$, d, e, $\boxed{de}$	$\boxed{ab}$, $\boxed{ac}$, ad, ae, bc, bd, be, cd, ce, $\boxed{ade}$, bde, cde, bcd, bce, $\boxed{bcde}$	abc, abd, abe, acd, ace, abcd, abde, acde, abce, $\boxed{abcde}$	Max elements in each rank (called "flats")

$$\chi_M(q) = q^3 + (-5+1)q^2 + (10-5)q + (-5+4-1)$$
$$= 1 \cdot q^3 - 4q^2 + 5q - 2$$
$$= \sum_{i\geq 0} (-1)^i \underbrace{f_{i-1}(BC(M))}_{\text{f-vector of broken circuit complex}} q^{\text{rank}(M)-i}$$

(independent of choice of order !!!)

Theorem

- For any matroid M and any ordering of the grand set,
$$\chi_M(q) = \sum_{i\geq 0} (-1)^i f_{i-1}(BC(M))\, q^{\text{rank}(M)-i}$$
- For any graphical matroid that stems out of G, χ_M is the chromatic polynomial of G, if G is loopless.
- For any linear matroid, consider the central hyperplane arrangement $\mathcal{A}$ of the hyperplanes normal to the vectors of the matroid. Then, $\chi_M = \chi_{\mathcal{A}}$.

Definition

Let Δ be a simplicial complex. Its <u>h-vector</u> is a "compact" version of the f-vector, given by

$$\sum_{k=0}^{d} f_{k-1}(q-1)^{d-k} = \sum_{k=0}^{d} h_{k-1} q^{d-k}.$$

Examples

$M =$ 

Δ	$BC(M)$	independence
$f(\Delta)$	$(1,4,5,2)$	$(1,5,9,5)$
$h(\Delta)$	$(1,1,0,0)$	$(1,2,2,0)$

The Tutte polynomial

Many invariants are evaluations of a single invariant:

Definition

The Tutte polynomial of a matroid $M=(E,\mathcal{I})$ is

$$T_M(x,y) = \sum_{A\subseteq E} (x-1)^{\operatorname{rank}(M)-\operatorname{rank}(A)} (y-1)^{|A|-\operatorname{rank}(A)}.$$

Theorem

For any matroid M, let $r := \text{rank}(M)$. Then,

- the chromatic polynomial of M is $\chi_M(q) = (-1)^r T_M(1-q, 0)$ (1)
- the broken circuit complex satisfies

$$\sum_{i\geq 0} f_{i-1}(\Delta)\, q^{r-i} = T_M(q+1, 0) \qquad (2)$$

and

$$\sum_{i\geq 0} h_{i-1}(\Delta)\, q^{r-i} = T_M(q, 0) \qquad (3)$$

- the independence complex satisfies

$$\sum_{i\geq 0} f_{i-1}(\Delta)\, q^{r-i} = T_M(q+1, 1) \qquad (4)$$

and

$$\sum_{i\geq 0} h_{i-1}(\Delta)\, q^{r-i} = T_M(q, 1). \qquad (5)$$

Proof

(1)
$$(-1)^r T_M(1-q, 0) = (-1)^r \sum_{A\subseteq E} (-q)^{r-\text{rank}(A)} (-1)^{|A|-\text{rank}(A)}$$
$$= \sum_{A\subseteq E} (-1)^{|A|} q^{r-\text{rank}(A)}$$
$$= \chi_M(q) \quad \text{(by definition)}$$

(2) Recall that $\chi_M(q) = \sum_{i\geq 0} (-1)^i f_{i-1}(BC(M))\, q^{r-i}$

Hence,
$$\sum_{i\geq 0} (-1)^i f_{i-1}(BC(M))\, q^{r-i} = \chi_M(q)$$
$$= (-1)^r T_M(1-q, 0) \quad \text{by (1)}$$
$$= (-1)^r \sum_{A\subseteq E} (-q)^{r-\text{rank}(A)} (-1)^{|A|-\text{rank}(A)}$$

$$= \sum_{A \subseteq E} (-1)^{|A|} q^{r-\mathrm{rank}(A)}$$

$$= \sum_{i \geq 0} \sum_{\substack{A \subseteq E \\ \mathrm{rank}(A)=i}} (-1)^{|A|} q^{r-i}$$

$$= \sum_{i \geq 0} (-1)^i \underbrace{\left(\sum_{\substack{A \subseteq E \\ \mathrm{rank}(A)=i}} (-1)^{|A|-i} \right)}_{= f_{i-1}(BC(M)), \text{ by comparison with line (1)}} q^{r-i}$$

On the other hand

$$T_M(1+q, 0) = \sum_{A \subseteq E} (-1)^{|A|-\mathrm{rank}(A)} q^{r-\mathrm{rank}(A)}$$

$$= \sum_{i \geq 0} \sum_{\substack{A \subseteq E \\ \mathrm{rank}(A)=i}} (-1)^{|A|-i} q^{r-i}$$

$$= \sum_{i \geq 0} f_{i-1}(BC(M)) \, q^{r-i}$$

(3) From (2), we have

$$T_M(1+q, 0) = \sum_{i \geq 0} f_{i-1}(BC(M)) \, q^{r-i}.$$

set $q = q' - 1$

$$\Rightarrow \sum_{i \geq 0} f_{i-1}(BC(M)) (q'-1)^{r-i}$$

$$= \sum_{i \geq 0} h_{i-1}(BC(M)) \, q'^{\,r-i} \qquad \text{definition of } h_{i-1}$$

$$= \sum_{i \geq 0} h_{i-1}(BC(M)) (q+1)^{r-i}.$$

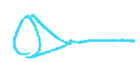
Therefore, $T_M(q, 0) = \sum_{i \geq 0} h_{i-1}(BC(M)) \, q^{r-i}$

(4) $$T_M(q+1,1) = \sum_{A\subseteq E} q^{r-\mathrm{rank}(A)} \underbrace{(1-1)^{|A|-\mathrm{rank}(A)}}_{=\begin{cases}1 & \text{if } |A|=\mathrm{rank}(A) \text{ (i.e. } A\in\mathcal{I}) \\ 0 & \text{otherwise}\end{cases}}$$

$$= \sum_{A\in\mathcal{I}} q^{r-\mathrm{rank}(A)}$$

$$= \sum_{i\geq 0} \underbrace{f_{i-1}(\Delta)}_{\#\{A\in\mathcal{I},\ |A|=i\}} q^{r-i},$$ where Δ is the independence complex.

(5) Proceed exactly like (3), using (4).

Example $M =$ 

$$T_M(x,y) = (x-1)^3 + (5+(y-1))(x-1)^2 + (9+5(y-1)+(y-1)^2)(x-1) + (5+4(y-1)+(y-1)^2)$$

$$= x^3 + x^2y + xy^2 + x^2 + xy$$

Consequences:

$$\chi_M(q) = (-1)^3 \cdot T_M(1-q,0) = (-1)^3((1-q)^3 + (1-q)^2)$$
$$= q^3 - 4q^2 + 5q - 2.$$

f-vector of $BC(M)$: $$T_M(q+1,0) = (q+1)^3 + (q+1)^2$$
$$= q^3 + 4q^2 + 5q + 2$$
$$= \sum_{i\geq 0} f_{i-1}(BC(M))q^{3-i}$$

$\Rightarrow$ f-vector$(BC(M)) = (1,4,5,2)$

h-vector of $BC(M)$: $T_M(q,0) = q^3 + q^2 = \sum_{i\geq 0} h_{i-1}(BC(M))\, q^{3-i}$

$\Rightarrow$ h-vector $(BC(M)) = (1,1,0,0)$

f-vector of $\mathcal{I}$: $$T_M(q+1,1) = (q+1)^3 + 2(q+1)^2 + 2(q+1)$$
$$= q^3 + 5q^2 + 9q + 5$$

$\Rightarrow$ f-vector of $\mathcal{I}$: $(1,5,9,5)$

h-vector of I: $T_M(q,1) = q^3 + 2q^2 + 2q$

$\Rightarrow$ h-vector of I: $(1,2,2,0)$

Five sequences of numbers from matroids.

Consider an integer sequence $a = (a_1, a_2, \dots, a_n)$. It is said to be

- unimodal if there exists $k \in \{1, \dots, n\}$ such that

$$a_1 \leq \dots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \dots \geq a_n$$

- log-concave if, for all $k \in \{2, \dots, n-1\}$, $a_k^2 \geq a_{k-1} a_{k+1}$
- flawless if $a_i a_j \neq 0 \Rightarrow a_k \neq 0$ for all $1 \leq i \leq k \leq j \leq n$.
- top-heavy if $a_i \leq a_{d-i}$ for all $0 \leq i \leq \frac{d}{2}$, where d is the largest index such that $a_d \neq 0$.

Note that log-concave $\Rightarrow$ (unimodal $\Leftrightarrow$ flawless).

Conjectures (several people, between 1968 and 2003)

The following sequences are unimodal, log-concave and top heavy:

a) The f-vector of the independence complex

b) The h-vector of the independence complex

c) The f-vector of the broken circuit complex

d) The h-vector of the broken circuit complex

e) The number of maximal subsets of each rank (equivalently, the intersections at each rank of the intersection poset if the arrangement is linear, or the number of flats)

Theorems (2012-2022; Huh, in different papers, with several collaborators: Adiprasito, Ardila, Braden, Denham, Katz, Matherne, Proudfoot, Wang)

- The sequences a), b), c), d) are unimodal, log-concave, top-heavy and flawless.
- The sequence e) is top-heavy and flawless.

Conjectures (open).

[Rota, 1971] Sequence e) is unimodal.

[Mason, 1972] Sequence e) is log-concave.

References: [AM23], [Eur23]