Math 108 - Geometric combinatorics Nadia Lafrenière

Cluster algebras from triangulations 2/1/2023

Definition

A cluster algebra is a commutative ring whose variables come in clusters, and are called cluster variables.

All the clusters can be obtained from a seed through a sequence of mutations.

"Type A" cluster algebras: arising from triangulation of an $(m+3)$-gon.

Clusters: triangulations.

Cluster variables: Label of the edges of the triangulation.

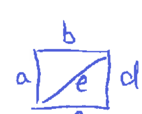
The boundary edges are frozen variables, and it is common to give them label 1.

Seed: a distinguished cluster.

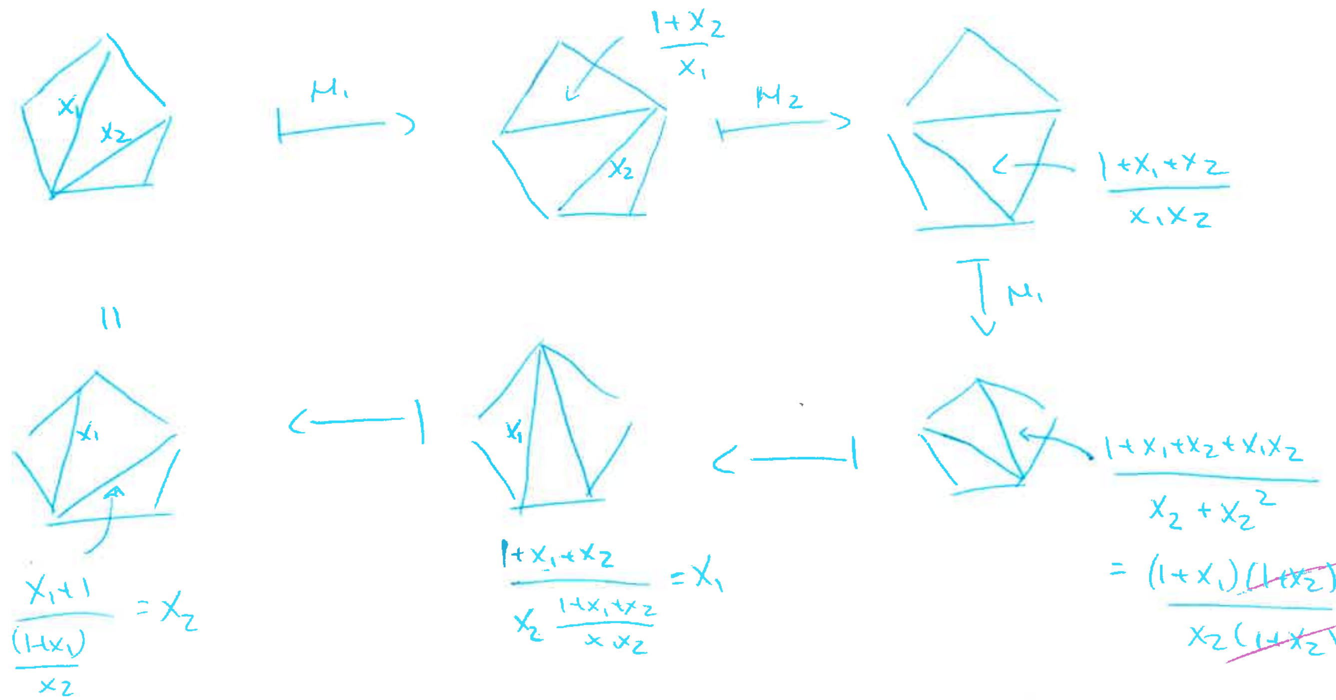
The fan

Mutation: a flip. Given an edge e of the triangulation, the label of the edge that replaces e is

$$\frac{ad+bc}{e}$$

where e is the diagonal of the quadrilateral 

Example: A_2 (triangulations of a pentagon)



5 clusters (triangulations):

$\{x_1, x_2\}$

$\{x_2, \frac{1+x_2}{x_1}\}$ 5 variables total

$\{\frac{1+x_1+x_2}{x_1x_2}, \frac{1+x_2}{x_1}\}$

$\{\frac{1+x_1}{x_2}, \frac{1+x_1+x_2}{x_1x_2}\}$

$\{x_1, \frac{1+x_1}{x_2}\}$

Observations:

(i) Well defined process: A given diagonal always receives the same label, independently of the sequence of flips to get there.

(ii) Laurent phenomenon: labels are all Laurent polynomials (i.e. a polynomial divided by a monomial).

(iii) Total positivity: The coefficients of the numerator are all nonnegative.

Total positivity might seem "obvious" here because there is no subtraction in the formula, but quotienting can create negative signs, such as in the following expression:

$$\frac{(x^3+1)}{(x+1)} = x^2 - x + 1.$$

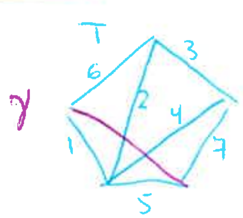
To make those observations a theorem, we will need an alternative formula for finding cluster variables.

Definition

Let T be a triangulation and let γ be an edge not in T. Let a and b be the endpoints of γ. A (T,γ)-path is a simple path of T (meaning no edge is repeated) $(e_1, \cdots, e_k)$ from a to b satisfying

- $e_1, \ldots, e_k$ are in T
- k (the length of the path) is odd.
- $e_2, e_4, \cdots, e_{2i}, \cdots$ intersect γ. Travelling from a to b on γ, we intersect $e_2, e_4, \ldots, e_{2i}, \ldots$ in this same order.

Example



$e_1\ e_2\ e_3\ e_4\ e_5$ is a (T,γ)-path.

$e_1 e_4\ e_3\ e_2\ e_5$ is not, because of the

All (T,γ)-paths: $e_6\ e_2\ e_5$

$e_1\ e_4\ e_7$

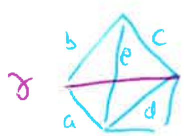
$e_1\ e_2 e_3\ e_4 e_5$

Theorem (Fomin and Zelevinsky, 2002, without proof; Schiffler and Thomas, 2009)

The cluster variable x_γ is given as

$$\sum_{\substack{(e_1,\dots,e_k)\text{ a}\\(T,\gamma)\text{-path}}} \frac{\prod_{i=1,\ i\text{ odd}}^{k} x_{e_i}}{\prod_{i=1,\ i\text{ even}}^{k} x_{e_i}}$$

The case where T and γ cross only once follows easily from the exchange condition:



$$x_\gamma = \frac{ac + bd}{e}$$

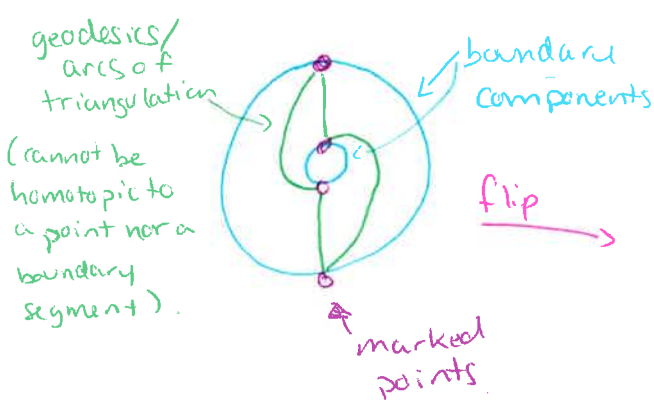
(T,γ)-paths:
bed
aec

$$x_\gamma = \frac{bd}{e} + \frac{ac}{e}.$$

Other cases are proven by induction the number of crossing of T and γ.

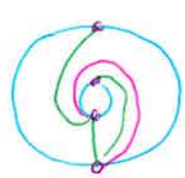
"Triangulations" of other surfaces.

Consider a more general type of surfaces. On each boundary, we draw at least one marked point (like vertices).



A triangulation is a decomposition into regions such that each region is bounded by 3 arcs.

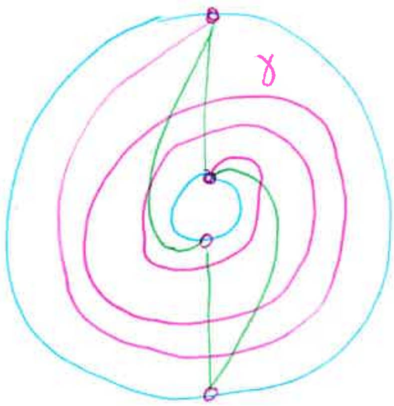
flip



Theorem

For cluster algebras arising from marked surfaces, the cluster variables are also given by the Schiffler - Thomas formula, and therefore all well-defined, totally positive Laurent polynomial. However, there are infinitely many of them.

Example



γ is also an arc of the triangulation. It is not homotopic to any arc with fewer than 10 crossings.

References: S. Fomin and N. Reading. Root systems and generalized associahedra in [MRS 07], 94.

R. Schiffler and H. Thomas. On Cluster algebras arising from unpunctured surfaces, 2009.