Math 108 - Geometric combinatorics — Nadia Lafrenière

Cluster algebras and triangulations — 1/30/2023

Recall that a <u>triangulation</u> of a polygon is a decomposition of it into triangles that adds no new vertices, and such that no edges cross.

Enumeration of triangulations:

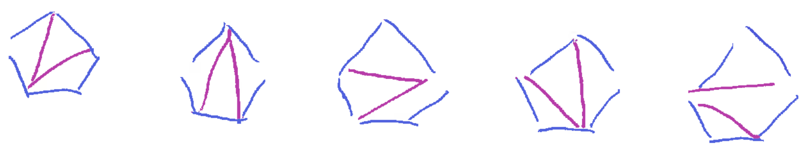
$n=3$

1 triangulation

$n=4$

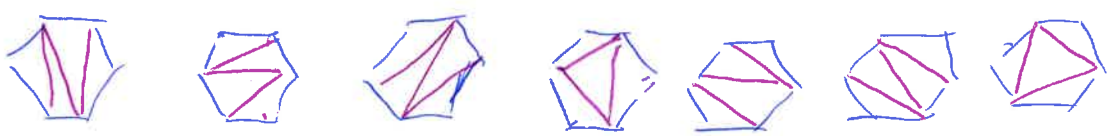
2 triangulations

$n=5$



5 triangulations

$n=6$



14 triangulations.

<u>Theorem</u>

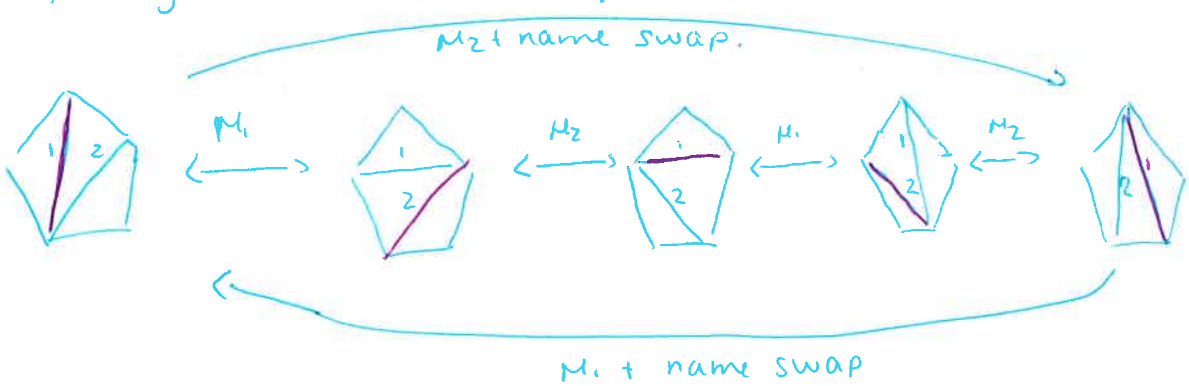
The number of triangulations of a convex n-sided polygons is the $(n-2)$-th Catalan number:

$$C_{n-2} = \frac{1}{n-1}\binom{2n-4}{n-2}$$

Given an n-sided polygon, a triangulation of it and an edge e of the triangulation (that is not an edge of the polygon), the <u>flip of e</u> is the triangulation obtained by removing e and replacing it with the unique edge that can be added:

Example

From a triangulation of the pentagon, one can get all other triangulations by a sequence of flips.

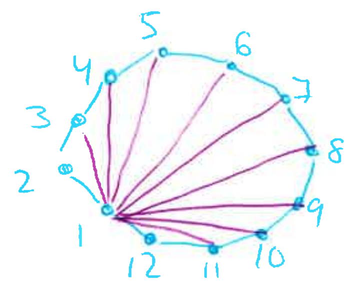


<u>Observation</u>: the flip of a given edge is an involution.

<u>Proposition</u>: let P be an n-sided polygon, and let T and T' be two triangulations of P. Then, there exists a sequence of flips that changes T into T'.

To prove this proposition, we need to define a distinguished triangulation.

Let P be an n-sided polygon with vertices numbered $\{1, 2, \dots, n\}$. The <u>fan</u> is the triangulation with edges $\{13, 14, \dots, 1(n-1)\}$.



The fan on a dodecagon.

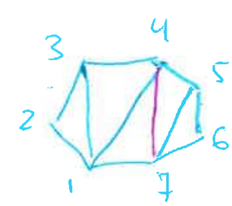
Lemma

Given a triangulation T, there is a sequence of flips from T to the fan.

Proof

We first observe that the fan is the triangulation that minimizes the sum of the endpoints over the edges.

<u>Claim</u>: If T is not the fan, there exists an edge jl that, if removed, creates an empty quadrilateral with vertices $i<j<k<l$.



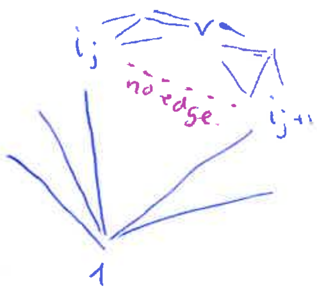
Assuming the claim is true. Then, choose an edge as in the claim as long as T is not the fan. Changing jl for ik yields a valid triangulation and allows us to get closer to the fan by decreasing the sum of endpoints of the edges.

We must prove the claim.

Assume T is not the fan:

Write $12, 1i_1, \ldots, 1i_k, 1n$ for all the edges, including the bondary, that are incident to vertex 1. Since T is not the fan, there exists $v \in \{3, \ldots, n-1\}$ such that $1v$ is not in the triangulation, and let $i_j < v < i_{j+1}$, and consider $1 < i_j < v < i_{j+1}$. If $i_j i_{j+1}$ is an edge of T, we are done.

Otherwise, we are in the following situation.



There is a triangle made of the sides $1i_j$ and $1i_{j+1}$. Therefore, $i_j i_{j+1}$ must be an edge, so it is the case above. (with $1 < i_j < v < i_{j+1}$ satisfying the condition of the claim).

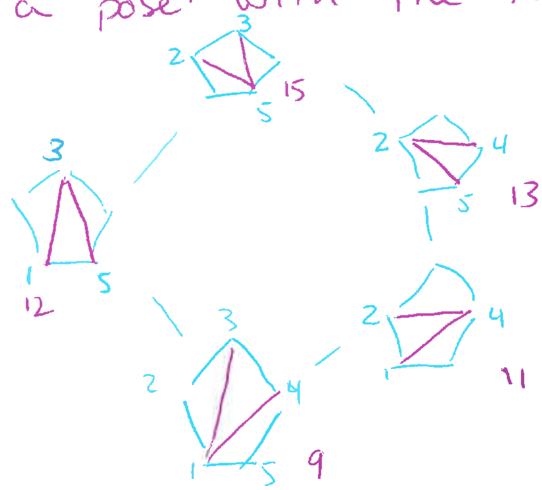
Therefore, the claim is true, which means that we can always decrease the sum of the endpoints until we get to the claim. This proves the lemma. $\blacksquare$

Proof of the proposition

To change T into T', consider the sequence of flips to get from T to the fan, and let it be followed by the sequence of flips to get from T' to the fan, in reverse order. This gives a sequence of flips to get from T to T'!

Remark: A sequence of flips is said to be increasing (resp. decreasing) if the sum of the endpoints is increasing (resp. decreasing) at each step of the sequence.

One gets a poset with the fan as minimal element:



The Tamari Lattice!

Overview of next lecture:

- A cluster algebra is a commutative ring with variables (the cluster variables) given by a seed and an exchange relation
- An important class of cluster algebras has clusters (of cluster variables) given by triangulations of a polygon, and the exchange relation given by flips

Reference: [Wil 14], §2

- Fomin, Shapiro, Thurston. Cluster algebras and triangulated surfaces, part I: Cluster complexes, 2008.