Math 108 - Geometric Combinatorics

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Characteristic polynomial and regions

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Theorem (Zaslavsky, 1975)

Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^n$.
Then, the number of regions of $\mathcal{A}$ is

$$r(\mathcal{A}) = (-1)^n \chi_{\mathcal{A}}(-1)$$

and the number of relatively bounded regions is

$$b(\mathcal{A}) = (-1)^{\operatorname{rank}(\mathcal{A})} \chi_{\mathcal{A}}(1).$$

To prove this result, we recall a few useful lemmas:

- (Generalized Sweep Method)

  Given a triple of arrangements $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$,

  - $r(\mathcal{A}) = r(\mathcal{A}') + r(\mathcal{A}'')$
  - $b(\mathcal{A}) = \begin{cases} b(\mathcal{A}') + b(\mathcal{A}'') & \text{if } \operatorname{rank}(\mathcal{A}) = \operatorname{rank}(\mathcal{A}') \\ 0 & \text{if } \operatorname{rank}(\mathcal{A}) = \operatorname{rank}(\mathcal{A}')+1 \end{cases}$

- (Deletion-restriction)

  Given a triple of arrangements,

  $$\chi_{\mathcal{A}}(t) = \chi_{\mathcal{A}'}(t) - \chi_{\mathcal{A}''}(t)$$

- (Rank from intersection poset).

  The rank of a hyperplane arrangement $\mathcal{A}$ is the height of its intersection poset $L(\mathcal{A})$.

  Therefore, $\operatorname{rank}(\mathcal{A}'') = \operatorname{rank}(\mathcal{A}) - 1$ for any triple of arrangement $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$.

Proof of Zaslavsky's theorem

We rewrite the statements as follows:

(1) $r(\mathcal{A}) = (-1)^n \chi_{\mathcal{A}}(-1) = |\chi_{\mathcal{A}}(-1)| = \sum_{X \in L(\mathcal{A})} |\mu(X)|$

(2) $b(\mathcal{A}) = (-1)^{\operatorname{rank}(\mathcal{A})} \cdot \chi_{\mathcal{A}}(1) = |\chi_{\mathcal{A}}(1)|$.

To prove (1), we proceed by induction on $\#\mathcal{A}$. (the # of hyperplanes)

- if $\mathcal{A} = \emptyset$, $r(\mathcal{A}) = 1$, and $\chi_{\mathcal{A}}(t) = t^n$.
- Otherwise, let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a triple of arrangements. We have the two recurrences:
  - $r(\mathcal{A}) = r(\mathcal{A}') + r(\mathcal{A}'')$
  - $(-1)^n \chi_{\mathcal{A}}(-1) = (-1)^n (\chi_{\mathcal{A}'}(-1) - \chi_{\mathcal{A}''}(-1))$.

$$= (-1)^n \chi_{\mathcal{A}'}(-1) + (-1)^{n-1} \chi_{\mathcal{A}''}(-1)$$

Induction hypothesis because $\#\mathcal{A}' = n-1$, $\#\mathcal{A}'' \leq n-1$

$$\stackrel{\text{Induction hypothesis}}{=} r(\mathcal{A}') + \underbrace{r(\mathcal{A}'')}_{\text{because } \dim(\mathcal{A}'') = \dim(H_0) = n-1}$$

Since $r(\mathcal{A})$ and $(-1)^n \chi_{\mathcal{A}}(-1)$ have the same value when $\mathcal{A} = \emptyset$ and satisfy the same recurrence, eq. (1) is proven.

Proof of (2)

If $\mathcal{A} = \emptyset$, then the unique region is relatively bounded, as its essentialization lives in $\mathbb{R}^0 = \{(0,0,\dots,0)\}$. Hence, $b(\emptyset) = 1$, and $(-1)^0 \cdot 1^0 = 1$. Hence, $b(\emptyset) = (-1)^{\operatorname{rank}(\emptyset)} \chi_{\emptyset}(1)$.

Otherwise, let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a triple of arrangements

- If $\operatorname{rank}(\mathcal{A}) = \operatorname{rank}(\mathcal{A}')$, then $\operatorname{rank}(\mathcal{A}'') = \operatorname{rank}(\mathcal{A}) - 1$ and

$$(-1)^{\operatorname{rank}(\mathcal{A})} \cdot \chi_{\mathcal{A}}(1) = (-1)^{\operatorname{rank}(\mathcal{A}')} \cdot \chi_{\mathcal{A}'}(1) - (-1)^{\operatorname{rank}(\mathcal{A}'')+1} \chi_{\mathcal{A}''}(1)$$

$$\stackrel{\text{induction hypothesis}}{=} b(\mathcal{A}') + b(\mathcal{A}'')$$

$$\stackrel{\text{Generalized sweep method}}{=} b(\mathcal{A})$$

- If $\operatorname{rank}(\mathcal{A}) = \operatorname{rank}(\mathcal{A}') + 1$, then $L(\mathcal{A}') \cong L(\mathcal{A}'')$

Following the deletion-restriction recurrence,

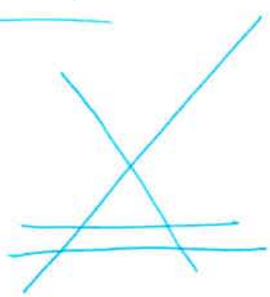
$$\begin{aligned}
\chi_{\mathcal{A}}(1) &= \chi_{\mathcal{A}'}(1) - \chi_{\mathcal{A}''}(1) \\
&= \sum_{x \in L(\mathcal{A}')} \mu(x) - \underbrace{\sum_{x \in L(\mathcal{A}'')} \mu(x)}_{= \sum_{x \in L(\mathcal{A}')} \mu(x) \text{ since } L(\mathcal{A}') \cong L(\mathcal{A}'')} \\
&= 0 = b(\mathcal{A}).
\end{aligned}$$

We have thus proven (1) and (2). ∎

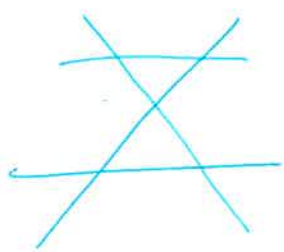
Corollary

Let $\mathcal{A}$ be a real hyperplane arrangement. Then, the number of (relatively bounded) regions ($r(\mathcal{A})$ and $b(\mathcal{A})$) depends only on $L(\mathcal{A})$.

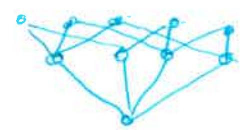
Example



and



have intersection poset



and therefore have the same number of regions (10) and bounded regions (2).

Example

Counting regions in the boolean arrangement $\mathcal{A}_n \subseteq \mathbb{R}^n$.

$$\chi_{\mathcal{A}_n}(t) = \sum_{x \in L(\mathcal{A})} \mu(x)\, t^{\dim(x)}$$

$$= \sum_{k=0}^{n} \sum_{\substack{x \in L(\mathcal{A}) \\ x \text{ has rank } k}} \underbrace{\mu(x)}_{(-1)^k}\, t^{\overbrace{\dim(x)}^{n-k}}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^k\, t^{n-k}$$

$$= (t-1)^n$$

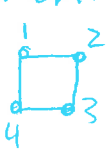
Therefore, $r(\mathcal{A}_n) = |\chi_{\mathcal{A}_n}(-1)| = 2^n$ and $b(\mathcal{A}_n) = |\chi_{\mathcal{A}_n}(1)| = 0^n = 0$.

Graphical arrangements

Let G be a graph on the vertices $[n]$, and let E be its set of edges. The graphical arrangement $\mathcal{A}_G$ has hyperplanes

$$\{ H_{ij} = \{v \in \mathbb{R}^n : v_i = v_j\},\ \{i,j\} \in E \}.$$

Examples:

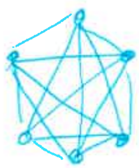
- $\mathcal{A}_{K_n}$, where K_n is the complete graph, is the braid arrangement.
- Let $G =$ . Then $\mathcal{A}_G = \{H_{12}, H_{23}, H_{34}, H_{14}\}$.

A <u>proper coloring</u> of a graph G is a map $\kappa : [n] \to [k]$, where the numbers $[k]$ are called "colors", such that $\kappa(i) \neq \kappa(j)$ for any $\{i,j\} \in E$.

The <u>chromatic polynomial</u> of G, $\chi_G(t)$, is the polynomial that counts the number of proper colorings of G with t colors.

Examples

$G=$  has chromatic polynomial $t(t-1)^3$.

$K_6=$  has chromatic polynomial $t(t-1)(t-2)(t-3)(t-4)(t-5)$

Theorem (Deletion-contraction)

Let G be a simple graph and let e be any edge in G.

Then,

$$\chi_G(t) = \chi_{G-\{e\}}(t) - \chi_{G\cdot e}(t),$$

where $G\cdot e$ is the graph obtained by merging the two endpoints of e and deleting multiple edges and the loop formed by e.

Example

χ  $(t) = \chi$  $(t) - \chi$  $(t) = t(t-1)^3 - t(t-1)(t-2)$

$= t(t-1)(t^2-3t+3)$

Theorem

Let $\mathcal{A}_G$ be a graphical arrangement. Then,

$$\chi_{\mathcal{A}_G} = \chi_G.$$

Idea of proof

The deletion-contraction and the deletion-restriction recurrences are "the same."

Counting regions for the braid arrangement:

Consider $\mathcal{B}_n$. Because $\mathcal{B}_n = \mathcal{A}_{K_n}$, where K_n is the complete graph,
$$\chi_{\mathcal{B}_n}(t) = \chi_{K_n}(t) = \prod_{i=0}^{n-1}(t-i)$$

From Zaslavsky's theorem:
$$r(\mathcal{B}_n) = |\chi_{\mathcal{B}_n}(-1)| = \left|\prod_{i=0}^{n-1}(-1-i)\right| = \prod_{i=1}^{n} i = n!$$

and
$$b(\mathcal{B}_n) = |\chi_{\mathcal{B}_n}(1)| = \prod_{i=0}^{n-1}(1-i) = \underbrace{(1-1)}_{0} \cdot \prod_{\substack{i=0 \\ i\neq 1}}^{n-1}(1-i) = 0.$$

Hence, there are $n!$ unbounded regions and 0 relatively bounded regions.

Reference: [Sta07], Lecture 2.