Math 108 - Geometric Combinatorics

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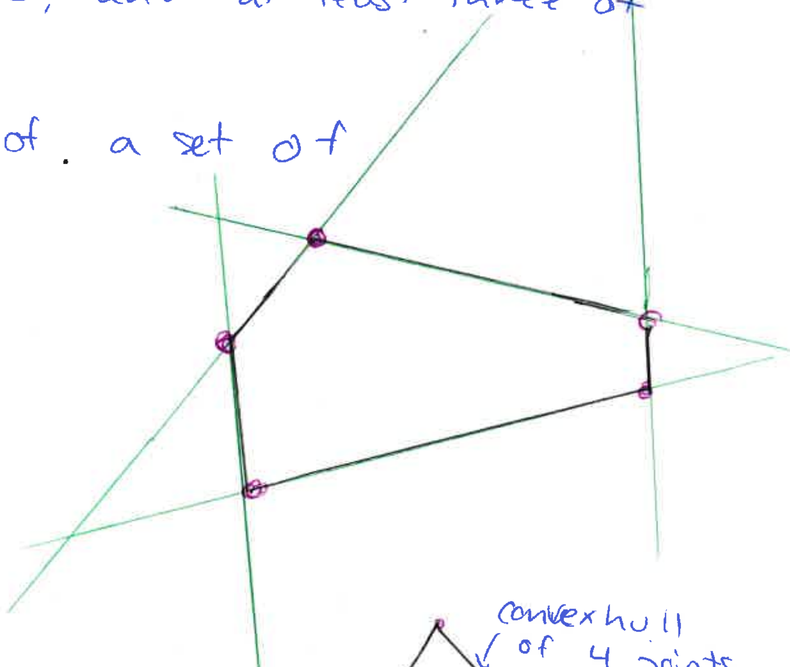
Course overview.

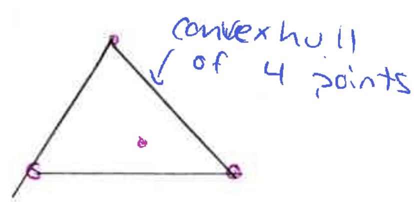
- Plane geometry : convex polygons.

What is a convex polygon in $\mathbb{R}^2$?

- The convex hull of a set of points (at least 3)
- A shape bonded by straight lines, that has as many vertices as edges, and at least three of them. + It is convex.
- It is a bonded region of a set of lines.



Possible counting questions:
- How many points are really needed to define it?
- What is its area?
- How many triangulations are there?
- How many bounded regions are there, for a set of lines?



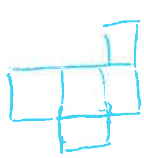
- Higher dimensions : convex polytopes.
  - Given a set of properties, can a polytope with all of them exist?

The geometry of geometries: matroids.

- Generalizes the notion of independence:
  - Linear independence of vectors
  - Acyclic graphs
- Gives new perspectives on hyperplane arrangements and simplicial complexes.
- We understand matroids from their invariants, and graph theory shows up in surprising places.

Polygons, again

- Given a convex polygon (of a fixed size), is it possible to tile the plane with infinitely many copies of it?
- Can that happen in more than one way?
- Dropping the convexity requirement, can a polyomino tile the plane?

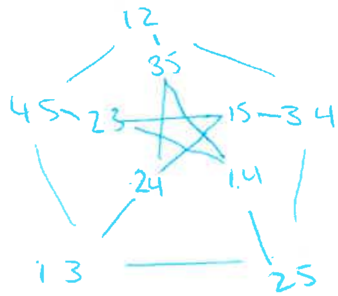


- allowing only translations?
- allowing rotations and translations?

Topological methods

A Kneser graph $KG_{n,k}$ has $\binom{n}{k}$ vertices, and two vertices (corresponding to subsets of size k of $[n]$) share an edge if they are disjoint.

Example



The Petersen graph is a Kneser Graph ($KG_{5,2}$) with chromatic number 3.

Theorem (Lovász 1978, conjectured by Kneser 1955)

The chromatic number of $KG_{n,k}$ is $n-2k+2$.

Theorem (Borsuk-Ulam)

Every continuous function from an n-sphere to an n-Euclidean space maps some pair of antipodal points to the same point.

Surprising fact: Borsuk-Ulam's Theorem is the main tool in Lovász's theorem!!!

The proof of the Kneser graph conjecture by Lovász is referred to as the beginning of topological combinatorics.