Math 25 Fall 2022

Midterm 0

Your name: ____

INSTRUCTIONS

You may begin the exam when ready.

Write your name in the space provided above, and check one box to indicate which section of the course you belong to.

Unless otherwise stated, you must justify your solutions to receive full credit. Work that is illegible may not be graded. Work that is scratched out will not be graded.

It is fine to leave you answer in a form such as $\ln(0.02)$ or $\sqrt{123412}$ or $(1341)^4(1231)^{-1}$. However, if an expression can be easily simplified (such as $e^{\ln(0.02)}$ or $\cos\pi$), you should simplify it.

The Honor Principle requires that you neither give nor receive any aid on this exam.

The exam has been created with the intended length of 50 minutes. It is intended to be the length of a standard midterm. This midterm is collected at the end of the X-hour.

Good luck!

Honor statement: I have neither given nor received any help on this exam, and I attest that all of the answers are my own work.

Signature: ____

Long Answer Questions

(1) (10 points) Use the division algorithm to compute gcd(101, 42).

Solution. We use the Euclidian algorithm. Executed in table format:

q_j	r_j	$q_j r_j$	a_j	b_j
	101	—	–	–
2	42	84	–	–
2	17	34	–	–
2	8	16	–	–
8	1	8	–	–
–	0	–	–	–

From the algorithm we see that $\gcd(101, 42) = 1$. □

(2) (10 points) Use induction to show that the following identity

$$1 - 2 + 3 - 4 + \ldots + (-1)^{n-1} \cdot n = \frac{n+1}{2}$$

holds for every odd positive integer n.

Solution. Note that every odd positive integer n is of the form $n = 2k-1$ for some $k \in \mathbb{N}$. We prove the result by induction on k. When $k = 1$, we have

$$1 = \frac{(2 \cdot 1 - 1) + 1}{2}.$$

So the base case holds. We now assume that the result is true for some $k \in \mathbb{N}$, and show that this implies the result for $k+1$. We have

$$\begin{aligned}
&1 - 2 + 3 - 4 + \ldots + (-1)^{2k-2} \cdot (2k-1) \\
&= 1 - 2 + 3 - 4 + \ldots + (-1)^{2k-4} \cdot (2k-3) + \left((-1)^{2k-3} \cdot (2k-2) + (-1)^{2k-2} \cdot (2k-1)\right) \\
&= \frac{2k-3+1}{2} + ((-1)^{2k-3} \cdot (2k-2) + (-1)^{2k-2} \cdot (2k-1)) \quad \text{Induction Hypothesis} \\
&= \frac{2k-3+1}{2} + (-(2k-2) + (2k-1)) \quad \text{because } 2k-2 \text{ is even} \\
&= \frac{2k-3+1}{2} + 1 \\
&= \frac{2k-1+1}{2}.
\end{aligned}$$

Thus, the result for k implies the result for $k+1$, and by induction the result is proven. □

(3) (10 points) Consider the following definition:

Definition. A natural number n is said to be *turquoise*[1] if it satisfies the following three properties:

- n is divisible by an odd prime,
- $n = a^2 + b^2$ for some positive natural numbers a, b, and
- $n^2 - 3n \geq 0$.

Find an example of a turquoise number, and explain why it is turquoise.

Solution. We claim that 5 is a turquoise number, which we check via the criteria.

- 5 is an odd prime, so divisible by itself.
- $5 = 1^2 + 2^2$.
- $5^2 - 3 \cdot 5 = 25 - 15 \geq 0$.

As all criteria are satisfied, 5 is "turquoise". □

[1] This is not a standard definition. It is a completely made-up concept for the sake of this midterm.

(4) (10 points) Is the following statement true:

Theorem (?)**.** *Let n, a, b be positive integers such that $n|a^2$ and $n|b^2$. Then $n|(a+b)$.*

If the statement is true, provide a proof. If not, provide a counter-example.

Solution. The statement is false. We consider $n = 4$, $a = 4$, and $b = 6$. Then

$$n \mid 4^2, 6^2 \qquad \text{but} \quad n \nmid (4+6).$$

$\square$

(5*) [2] (4 points) Let p be a prime and let a, b be integers. If

$$a^3 - 3a^2bx + 3ab^2x^2 - b^3x^3$$

is divisible by p for all $x \in \mathbb{Z}$, show that p divides both a and b.

Solution. Since the expression is divisible by p for all $x \in \mathbb{Z}$, we set $x = 0$ to see

$$p \mid (a^3 - 3a^2b \cdot 0 + 3ab^2 \cdot 0^2 - b^3 \cdot 0^3).$$

That is, $p \mid a^3$. Next, p is a prime and $p \mid a \cdot a \cdot a$, so by definition of being prime it divides either a or $a \cdot a$; either case implies $p \mid a$.

Next, we set $x = 1$ to see that

$$p \mid (a^3 - 3a^2b + 3ab^2 - b^3).$$

By factoring, we have that $p \mid (a - b)^3$. As before, we see that $p \mid a - b$. But, $p \mid a$, so p divides the integer linear combination $-(a - b) - a = b$. This proves the result. $\square$

[2] Star questions indicate increased difficulty/trickiness.

(This page is intentionally left blank in case you need extra space for any of the problems. **If you use this page for a particular problem, it is essential that you make a note on the page where the problem appears, indicating that your work is continued here.)**