Math 38 — Graph Theory Nadia Lafrenière

Coloring of planar graphs and maps

05/25/2022

One of the most interesting theorems in both graph theory and math history is the Four Color Theorem. We see two alternative versions of it, and the history of that theorem.

Six Color Theorem

Theorem

Every planar graph can be colored using six colors.

Lemma

If every subgraph of a graph G has a vertex of degree at most k, then G is $(k+1)$-colorable.

Proof of the lemma

By induction on the number n of vertices.

- The graph with one vertex is 1-colorable, and the only vertex has degree 0.
- Assume that every graph with at most n vertices such that all its subgraphs contain a vertex of degree at most k is $(k+1)$-colorable.
- We now look at a graph G with $n+1$ vertices, and we assume that every subgraph has a vertex of degree at most k. Let v be a vertex in G with degree at most k. Then, the induced subgraph $G-v$ has n vertices and satisfies the induction hypothesis; it is $(k+1)$-colorable. Since v has k neighbors, it can be colored with one of the $k+1$ colors.

Proof of the Six Color Theorem

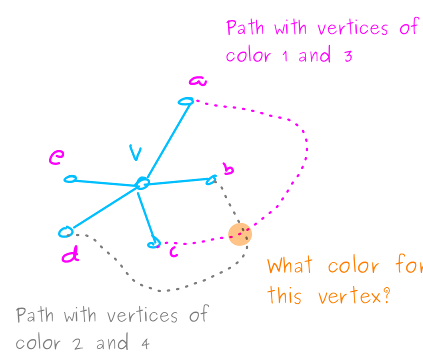
Recall that a planar graph with n vertices has at most $3n-6$ edges. That means that there is a vertex with degree at most 5 in every planar graph (and its subgraphs), because otherwise there would be at least $3n$ edges.

So every subgraph of G is planar and therefore contains a vertex of degree at most 5; using the lemma, it is 6-colorable.

Five Color Theorem

Five Color Theorem (Heawood, 1890)
Every planar graph is 5-colorable.

Sketch of proof (details in the textbook)
The proof uses strong induction on the number of vertices.
Base cases: If there are at most 5 vertices, the graph is 5-colorable.
Induction step: The graph has at least 6 vertices (and is planar). Using the same argument as for the Six Color Theorem, there is a vertex v of degree at most 5. Also, G−v satisfies the induction hypothesis, so it is 5-colorable. If v has fewer than 5 neighbors, or if two of the five neighbors of G have the same color, we can color v with the fifth color, and we are good.
So assume, by contradiction, that G−v is 5-colorable, but G is not.
Necessarily, that means v has neighbors with all different colors: label the five neighbors a,b,c,d and e, with colors respectively 1,2,3,4 and 5.
Also, since G is not 5-colorable, there is no way we can switch the colors of the neighbors of v to repeat the colors.
That means that, for any pair of colors i and j, the vertices in the neighborhood of G that have colors i and j are joined by a path alternating between vertices of color i and color j. Necessarily, these paths must cross at one point (details of the last argument are in the textbook).
The paths must cross at a vertex, because the graph is planar, but this vertex should have a color among 1 and 3 and among 2 and 4; a contradiction.
So if G−v is 5-colorable, G must also be 5-colorable.

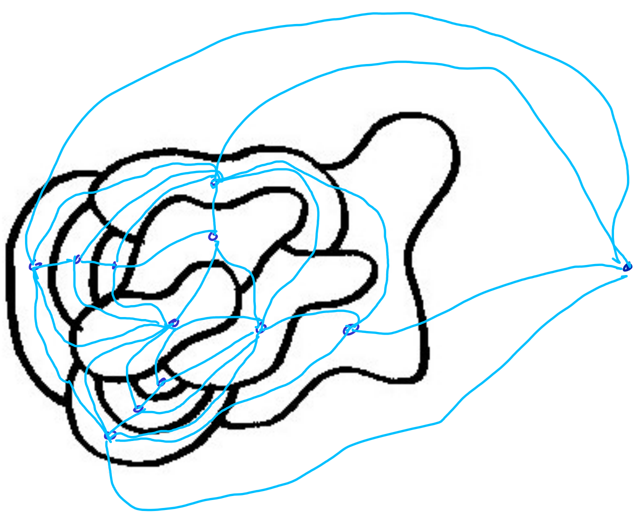


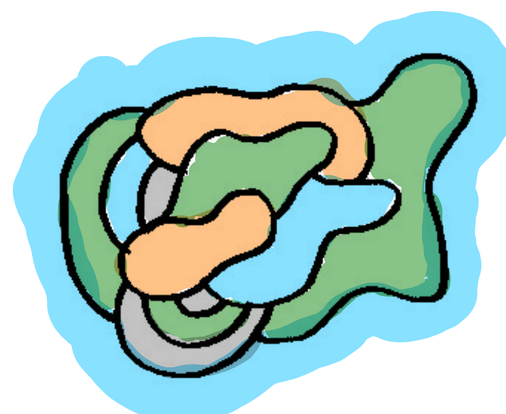
Remark

Maybe you are (sort of) convinced by this argument that if $G-v$ is five-colorable, then so is G, but what about if v had only 4 neighbors and $G-v$ was 4-colorable? Could we say that G is 4-colorable as well?

Four Color Theorem

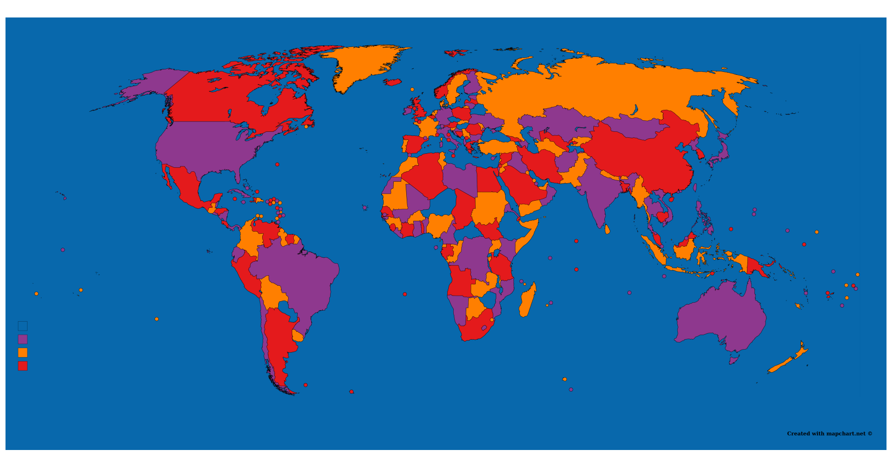
A map is a plane drawing. The dual graph of a map is a planar graph.





A coloring of a map is a coloring of its dual graph: two adjacent regions cannot be colored with the same color.

Theorem (Appel, Haken and their computer, 1976)
Every planar graph is 4-colorable. Every map is 4-colorable.

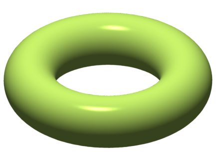


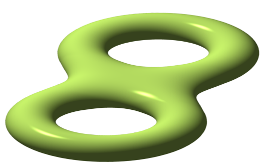
A bit of history on the Four Color Theorem

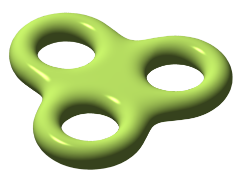
1852: Earliest known posing of the conjecture (in terms of map coloring)
1878: Popularization of the conjecture at a mathematical meeting
1879: First "proof", similar to the proof of the Five Color Theorem
1879–1976: Plenty of incomplete proofs
1976: Kenneth Appel and Wolfgang Haken prove the theorem using computer enumeration. They reduced the general case to 1,834 cases that the computer has to check. It takes over 1,000 hours to the computer to complete the proof, which cannot be read by a human being.
1996: Simplification of the proof to (only) 633 cases
2005: Computer verification of the proof
2022+? Can we imagine a proof that would be readable by a human?

Coloring on surfaces, in general

The genus of an orientable surface is its number of handles.


Genus 1

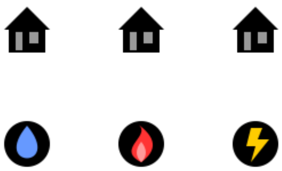

Genus 2


Genus 3

The plane and the sphere have genus 0

A graph is embeddable on a surface if it can be drawn on it without edges crossing.
A planar graph is embeddable on a plane.

Theorem
An orientable surface with genus g has Euler's characteristic $2-2g$.
That implies that a graph with n vertices and e edges that is embeddable on a surface of genus g satisfies $n-e+f=2-2g$, where f is the number of faces.



On a surface of genus 1, we can solve the 3-utility problem! (demonstration)

Theorem

If G is embeddable on a torus (genus 1), then G is 7-colorable.

Proof

We know:

- Euler's formule for the torus: $n-e+f = 2-2g = 0$.
- $2e$ = (sum of the length of the faces) $\geq 3f$.

This means that $3n-3e+3f = 0$. So, $3n-3e = -3f$, and $3n-3e \geq -2e$. So $3n \geq e$.

Thus, there is a vertex with degree at most 6 for every graph embeddable in the torus. This implies that every subgraph of G has a vertex of degree at most 6.

Using the lemma from the section for the six color theorem), that means that G is 7-colorable.

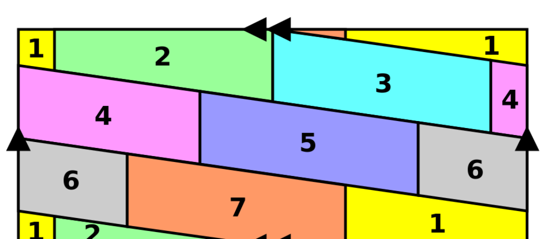
Theorem

There exist some graphs that are embeddable on the torus that have chromatic number 7.

Proof

K_7 can be embedded on the torus, as shown below. As every complete graph, its chromatic number is its number of vertices.

Source of the picture: Wikipedia



K_7

Reference: Douglas B. West. Introduction to graph theory, 2nd edition, 2001. Section 6.3