Math 38 – Graph Theory

Planarity

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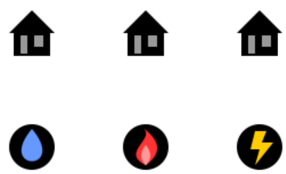
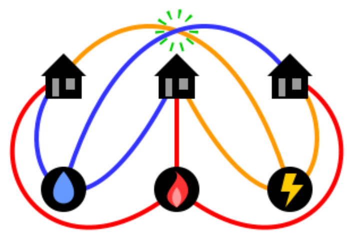
Is it possible to connect the three houses to the three utilites, without connection between two utilities nor connection between two houses? No crossing is allowed.



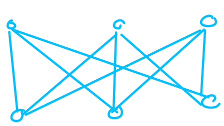


Pictures: Cmglee, on Wikipedia

(Answer is further in the notes).

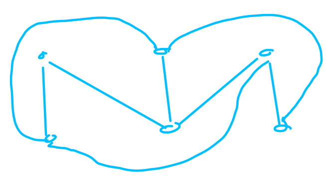
Planar graphs

Intuitively, a graph is planar if it can be drawn on a plane in a way such that no two edges cross.



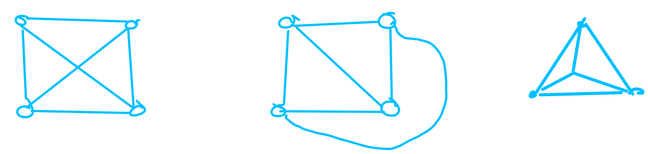
$K_{3,3}$

not planar



$K_{3,3}-e$

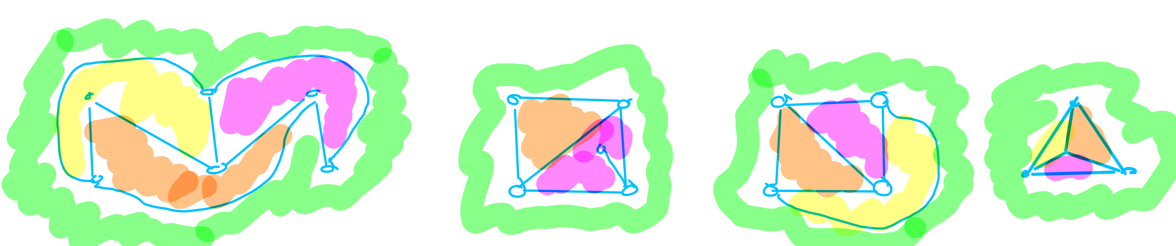
planar



K_4 is planar

A drawing of a graph is a geometric object, whereas the graph does not depend on the drawing. A plane graph (or planar embedding) is a drawing of a graph without crossing.

A face of a plane graph is a maximal region without a point of a curve for the drawing.



The outerface is the unbounded face. It is unique for a finite graph.

Dual graphs

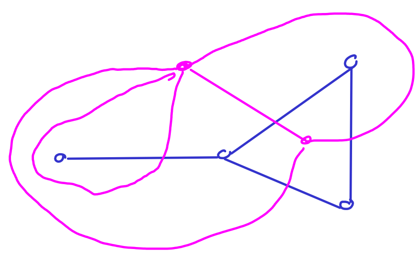
Given a plane graph G, we construct its dual, G^* in the following way:

- The faces of G become the vertices of G^*.
- The edges of G are connections between two adjacent faces. For every edge in G, connect the two faces in G^* by drawing an edge between the two vertices of G^*.

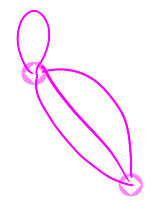
Remark

Even when G is a simple graph, G^* might have multiple edges or loops.

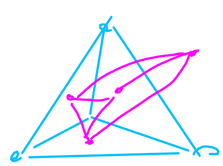
Example (the paw)



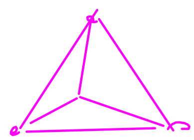
$\cong$



Example



$\cong$



$K_4^* \cong K_4$

Proposition

1 – The dual graph is planar.

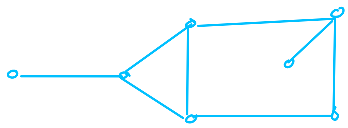
2 – $G^{**} \cong G$ if and only if G is connected.

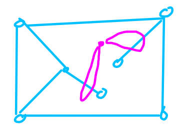
Proof of 1

Edges in the dual graph do not cross; they represent adjacency relationships. If they had to cross, that would mean two adjacent faces are in between two other adjacent faces D and F. If there is no way to draw an edge between D and F without crossing, then they cannot be adjacent.

Remark

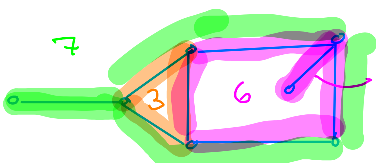
Two drawings of the same graph can have non-isomorphic duals.

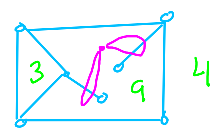




However, some properties will remain the same, like the number of faces, vertices and edges in the dual (if the graph is connected).

The length of a face in a plane graph is the total length of the closed walks in G bounding these faces (including the contour of the edges).





Observation

The sum of the lengths of the faces are equal.

Proposition

The sum of the length of the faces is twice the number of edges.

Proof

This is since every edge is neighboring either two faces, or it has both sides of the edges in the same face.

Alternative proof

The number of edges are the same in G and in its dual. The length of a face in G is the degree in G^*. By the sum of the degrees formula, twice the number of edges is the sum of the length of the faces.

Theorem

Let G be a plane graph. The following are equivalent.

(A) G is bipartite.

(B) Every face of G has even length.

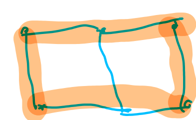
(C) G^* is Eulerian.

Proof

(A)=>(B) Since G is bipartite, all cycles have even length. The length of a face is the length of a closed walk. G cannot have faces of even length, since every closed walk (including the boundary of a face) contains an odd cycle, which would contradict the fact that the graph is bipartite.

(B)=>(A) If every face has even length, there cannot be odd cycles in G. Cycles are closed walks, so we will prove there is no odd closed walk in G. The boundary of one face cannot be an odd closed walk, by hypothesis. So an odd closed walk would need to be on the boundary of multiple faces.

Assume it exists. The number of edges on the boundary of these faces but not in the walk, counted with multiplicity, would need to be odd. However, since they are not in the walk (that is closed) they are counted for two faces, so their total number is even. Hence, the number of edges in the walk is also even. A contradiction.



(B)<=>(C) The dual graph is connected. The vertex degree in G^* is the length of the faces in G, so it is always even. A connected graph has only vertices of even degree if and only if it is Eulerian.

Euler's Formula

Theorem (Euler, 1758)

Let $G=(V,E)$ be a connected plane graph with f faces.

Then, $|V|-|E|+f=2$.

Key argument: this formula is also valid for convex solids, and the plane maps to the sphere (via stereographical projection).

Corollary

All drawings with no crossing have the same number of faces. The dual graphs of G all have the same number of vertices.

Remark

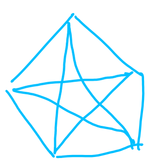
If you draw graphs on surfaces that are not the plane (like the torus), this changes. The number 2 here is called the Euler characteristic of the plane.

Corollary

If G is a simple planar graph with at least 3 vertices, then G has at most $3|V|-6$ edges. If it has no triangle, then $|E|\leq 2|V|-4$.

The key of the proof is that $2|E|$ is the sum of the length of the faces, and that each face uses at least 3 edges (4 if the graph has no triangle).

Example

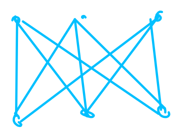


$|V| = 5$

$3|V|-6 = 9$

$|E| = 10$

Not planar



$|V| = 6$

Bipartite => Triangle-free

$2|V|-4 = 8$

$|E| = 9$

Not planar

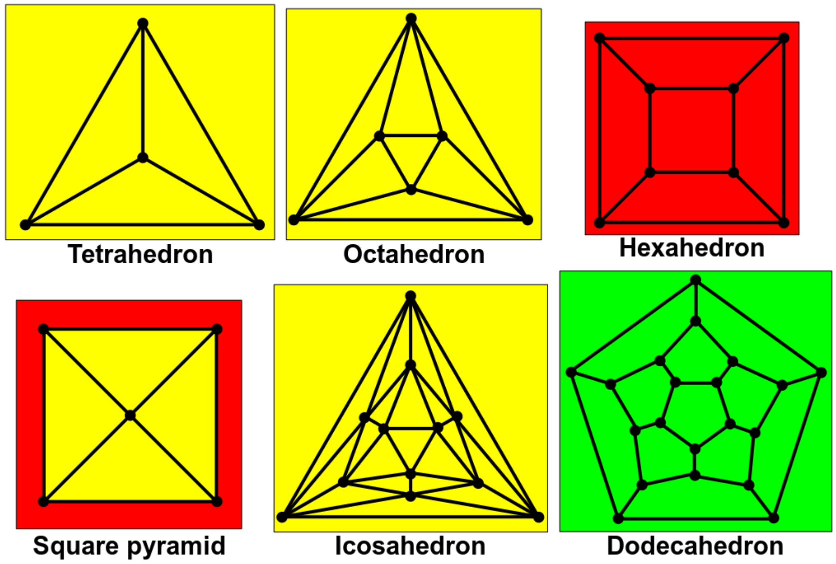
Connection to geometry

Convex polyhedra have Euler Characteric 2. This is true for Platonic solids.

Names	$\|V\|$	$\|E\|$	f	$\|V\|-\|E\|+f$
Cube	8	12	6	2
Tetrahedron	4	6	4	2
Octahedron	6	12	8	2
Icosahedron	12	30	20	2
Dodecahedron	20	30	12	2

Proposition

All platonic solids are planar graphs.



Reference: Douglas B. West. Introduction to graph theory, 2nd edition, 2001. Section 6.1