Math 38 – Graph Theory

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Chromatic polynomial for chordal graphs

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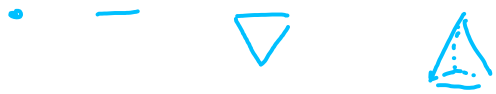
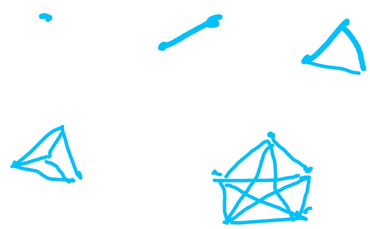
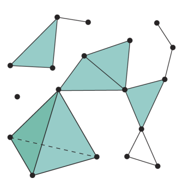
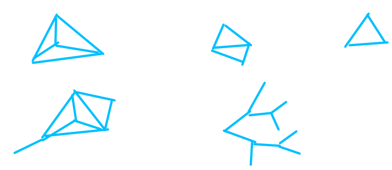
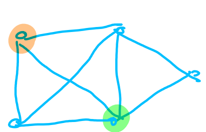
The goal of today's lecture is to give a shortcut for computing the chromatic polynomial of many graphs.

Geometry	Graph theory
Simplex (triangle, tetrahedron, etc.)	Complete graph
	
Simplicial complex (gluing simplices together) Every face of a simplicial complex is a simplex.	Chordal graph, i.e. graph that admit a simplicial elimination ordering.
	

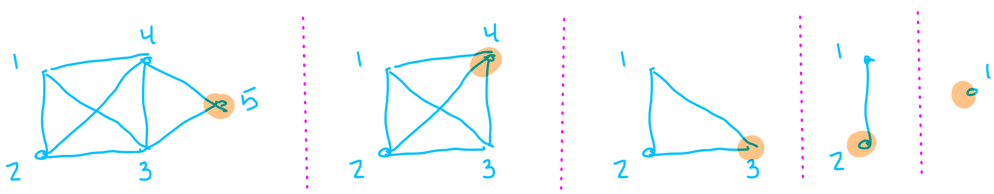
A vertex is <u>simplicial</u> if its neighbors form a clique.



Simplicial

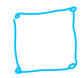
Not simplicial

A <u>simplicial elimination ordering</u> is an ordering of the vertices $v_n, \ldots, v_1$ such that the vertex v_i is simplicial in $G - \{v_n, \ldots, v_{i+1}\}$



Example

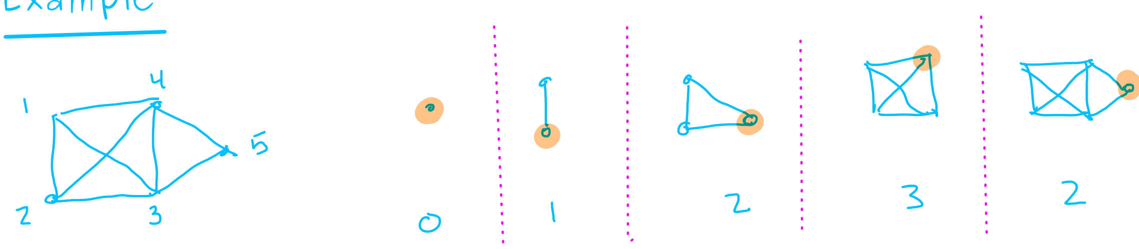
- For complete graphs, any ordering is a simplicial elimination ordering.
- For trees, an ordering starting at the leaves and going inwards is a simplicial elimination ordering.
- The cycle of length 4 has no simplicial ordering.



Proposition

The chromatic polynomial of a graph with simplicial ordering $v_n, \dots, v_1$ is the product of the $k-d'(v_i)$'s, where $d'(v_i)$ is the degree of v_i in the induced subgraph with vertices $\{v_1, \dots, v_i\}$.

Example



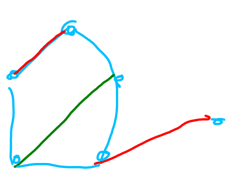
$$\chi(G;k) = k\,(k-1)(k-2)^2(k-3)$$

Sketch of proof

When it is added, v_i has $d'(v_i)$ neighbors, all colored with different colors, so there are $k-d'(v_i)$ options for coloring it. Doing this process, we count all the options for coloring the graph with k colors.

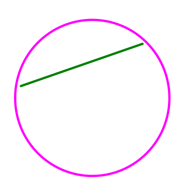
Chordal graphs

A chord of a cycle C is an edge not in C whose endpoints are in C.



Chord

Not a chord



A cycle is chordless if it has length at least 4 and no chord.

A graph is chordal if it is simple and it has no chordless cycle (as induced subgraphs).

Theorem (Dirac, 1961)
A simple graph has a simplicial elimination ordering if and only if it is a chordal graph.

Lemma (Voloshin, 1982, or Farber–Jamison, 1986)
Every chordal graph has a simplicial vertex.

The proof of the lemma is omitted. It is in the textbook as Lemma 5.3.16.

Proof of the theorem
$\Rightarrow$ (contrapositive) If it is not chordal, there is a chordless cycle C. In C, that has length at least 4, none of the vertices is simplicial, and no vertex can be the first one to be picked in C for the ordering.

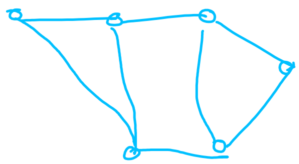
$\Leftarrow$ Using the lemma, we know that every chordal graph G has a simplicial vertex v. Delete v. Then $G-\{v\}$ is chordal, so we can apply the lemma again, creating an ordering.

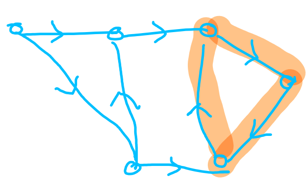
$\blacksquare$

Acyclic orientations

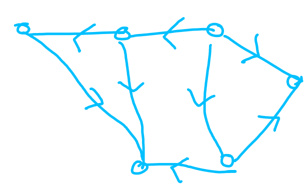
What is the meaning of $\chi(G;-1)$?

An orientation of a graph G is a digraph that has G as an underlying graph.





orientation



acyclic orientation

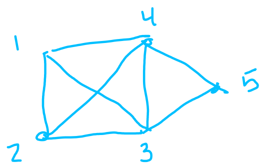
An orientation is acyclic if it has no cycle.

Theorem (Stanley, 1972)

The absolute value of $\chi(G;-1)$ is the number of acyclic orientations of G.

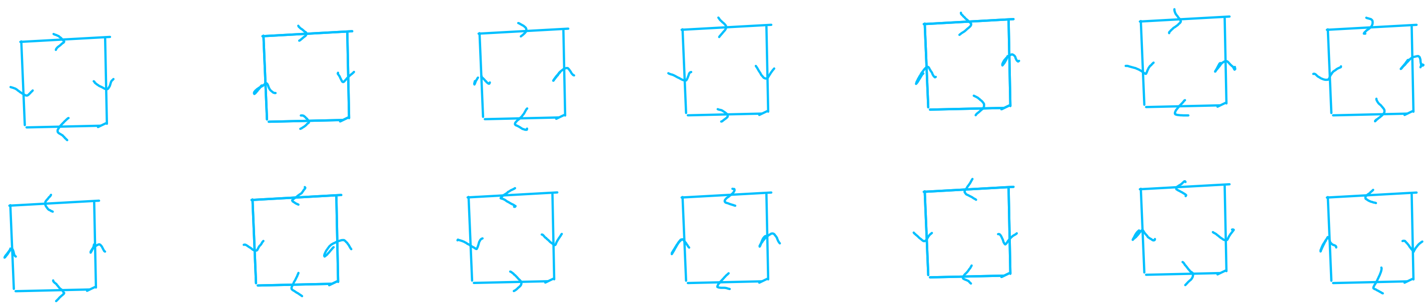
Proof is in the textbook, as the proof of Theorem 5.3.27.

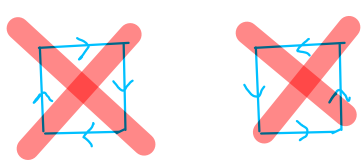
Examples



has chromatic polynomial $\chi(G;k) = k(k-1)(k-2)^2(k-3)$, so it has 72 acyclic orientations.

C_4 has chromatic polynomial $\chi(C_4;k) = k(k-1)(k^2-3k+3)$, and $\chi(C_4;-1)$ is 14.





Reference: Douglas B. West. Introduction to graph theory, 2nd edition, 2001. Section 5.3.