Chromatic polynomial

The problem: Counting the number of proper colorings of a graph G with k colors.

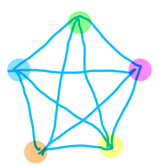
- If $\chi(G)>k$, then this number is 0.
- If $\chi(G)<k$, we must first choose which colors will appear, and then count the number of colorings with these colors.

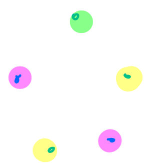
Notation

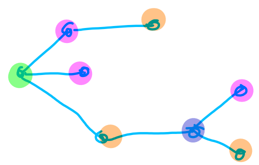
Given a graph G, the value $\chi(G;k)$ is the number of proper colorings of G with k colors.

Examples

- When G is the complete graph with n vertices, $\chi(G;k)=k(k-1)\dots(k-n+1)$.
  This number is also $\binom{k}{n}\, n!$
- When G is the graph with n vertices and no edge, $\chi(G;k)=k^n$.
- If T is a tree with n vertices, $\chi(T;k)=k(k-1)^{n-1}$.
- If P is a path with n vertices, $\chi(T;k)=k(k-1)^{n-1}$.







The chromatic polynomial of G is the polynomial $\chi: k\mapsto\chi(G;k)$.

Computation of the chromatic polynomial

A naive algorithm

Proposition

Let $p_r(G)$ be the number of partitions of the n vertices of G into r independent sets. Then, the chromatic polynomial of G is

$$\sum_{r=1}^{n} p_r(G)\binom{k}{r}r! = \sum_{r=1}^{n} p_r(G)\ k\cdot(k-1)\cdot(k-2)\cdots(k-r+1)$$

This is a unitary polynomial in the variable k of degree n, i.e. the leading term is k^n.

Proof

Given a coloring of G, the color classes partition the vertices of G into independent sets. If we have exactly r independent sets, there are $\binom{k}{r} r!$ ways of coloring them with r colors. Also, the number of color classes can be any number between 1 and n.

As for the maximum degree, it will happen when r is maximal. Since there is exactly 1 partition of the vertices into n independent sets (of size 1), the leading term is k^n.

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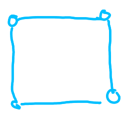
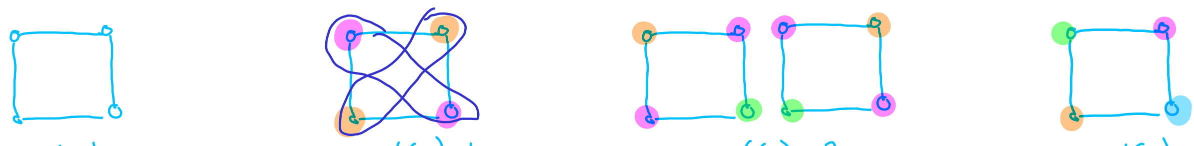
Example

Computing the chromatic polynomial of the cycle of length 4.

C_4 



$P_1(C_4)=0$ $P_2(C_4)=1$ $P_3(C_4)=2$ $P_4(C_4)=1$

Hence,

$$\chi(C_4;k) = \binom{k}{2}2! + 2\binom{k}{3}3! + \binom{k}{4}4!$$
$$= k(k-1)+2k(k-1)(k-2)+k(k-1)(k-2)(k-3)$$
$$= k(k-1)\ (1+2(k-2)+(k-2)(k-3))$$
$$= k(k-1)\ (k^2-3k+3)$$

Chromatic recurrence / Deletion-contraction

Theorem

If G is a simple graph and e is an edge of it, then
$\chi(G;k) = \chi(G-e;k) - \chi(G\cdot e;k)$, where $\cdot$ is the contraction operation used for counting spanning trees.

Notice that here, unlike with spanning trees, we may delete multiple edges at every step.

Example

$$\chi(C_4;k) = \chi(\underset{P_4}{\sqsupset};k) - \chi(\underset{K_3}{\triangleleft};k) = k(k-1)^3 - k(k-1)(k-2) = k(k-1)(k^2-3k+3)$$

Proof

The two endpoints of e cannot be colored with the same color. Hence, the number of proper colorings of G is the number of proper colorings without this edge, except those where the two endpoints are colored with the same color.

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Proposition

The chromatic polynomial of the cycle of size n is $(k-1)^n + (-1)^n(k-1)$.

Proof

We proceed by induction.

Base case: $n=2$. This is the complete graph (plus a multiple edge), so the chromatic polynomial is $k(k-1)$. This is equal to $(k-1)^2+(k-1)$.

Induction step: Assuming the chromatic polynomial of the cycle of length n is $(k-1)^n+(-1)^n(k-1)$, we want to prove that that of the cycle of length n+1 is $(k-1)^{n+1}+(-1)^{n+1}(k-1)$.

Using deletion contraction,

$$\begin{aligned}\chi(C_{n+1};k) &= \chi(P_{n+1};k) - \chi(C_n;k) \\ &\overset{\text{Induction hypothesis}}{=} k(k-1)^n - ((k-1)^n + (-1)^n(k-1)) \\ &= (k-1)^{n+1} + (-1)^{n+1}(k-1)\end{aligned}$$

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Proposition

Let $G=(V,E)$ be a graph. The second term of $\chi(G;k)$ is $-\#E\, k^{\#V-1}$.

Sketch of proof

We use the recurrence $\chi(G;k) = \chi(G-e;k) - \chi(G\cdot e;k)$ and induction on the number of edges. We do the proof on connected graphs.

When there is one edge, $\chi(G;k) = k^2-k$.

Induction step: G has one more edge than $G-e$, and the same number of vertices. By induction hypothesis, the second term of $\chi(G-e;k)$ is $(-\#E+1)k^{\#V-1}$; and $\chi(G;k)$ and $\chi(G-e;k)$ have the same leading term. Also, the leading term of $\chi(G\cdot e;k)$ is $k^{\#V-1}$.

Conclusion: The second term of $\chi(G;k)$ is $(-\#E+1-1)k^{\#V-1}$. $\blacksquare$

One last remark

Computing the chromatic polynomial looks much longer than computing the chromatic number, but the chromatic number is very hard to compute in general. If the graph does not permit to use a shortcut for computing its chromatic number, it is easier to compute its chromatic polynomial and to check which positive integer is not a root of the polynomial.

Reference: Douglas B. West. Introduction to graph theory, 2nd edition, 2001. Section 5.3