Math 38 — Graph Theory
Graph colorings

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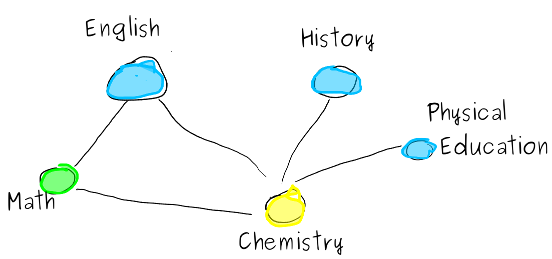
Recall from the very first lecture the following problem:

Scheduling and avoiding conflicts
My high school used to have a very long exam sessions at the end of the year, and there were still some conflicts. I wish the administrators knew graph theory...



Schedule:
1. History—English—PE
2. Chemistry
3. Math

Vertices: Subjects
Edges: If someone takes both subjects, i.e. eventual scheduling conflicts.

Intuitive definition:
A proper coloring of a graph is a partition of the vertices into independent sets.

Scheduling with no conflicts is equivalent to coloring.
If we want to use the minimum time, we should use as few colors as possible.

Definition
A k-coloring of a graph G is a labeling of the vertices using labels from a set of size k (called colors, even though the labels can be numbers, for example).

The vertices of one color form a color class.

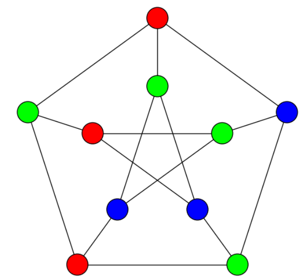
A coloring is proper if no two adjacent vertices have the same label.
A graph is k-colorable if it has a proper k-coloring.
The chromatic number $\chi(G)$ is the least k such that G is k-colorable.

In a proper coloring, every color class is an independent set. The chromatic number is the smallest number of independent sets that partition the vertices of a graph.

Example

The Petersen graph has chromatic number 3:

- It is not 2-colorable, because its vertices cannot be divided into two independent sets; it would otherwise be bipartite.
- It is 3-colorable, as shown on the right.



Notice that the chromatic number is an extremal problem: we need to show it is minimal and that a proper coloring exists.

Colorings for non-simple graphs

Graphs with loops do not admit proper colorings: a vertex that is incident to a loop could not be colored.

Every loopless graph can be colored: a trivial coloring where every vertex has a distinct color would work.

Multiple edges don't change anything to colorings, as two adjacent vertices cannot be colored the same color regardless of the number of edges between them.

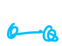
Optimality

A graph G is <u>k-chromatic</u> if $k=\chi(G)$; a proper k-coloring is then an optimal coloring.

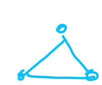
If $\chi(H)<\chi(G)=k$ for every subgraph H of G, then G is <u>k-critical</u> or color-critical.

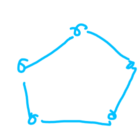
Examples

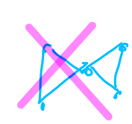
$k=1$ 

$k=2$ 

$k=3$: The 3-critical graphs are the smallest graphs that are not bipartite: these are the odd cycles.







Not 3-critical

No general characterization of 4-critical is known.

First bounds on the chromatic number

The clique number of a graph, written $\omega(G)$, is the maximum size of a clique in G. (Recall that a clique is a complete subgraph).

Also, recall that the independence number, $\alpha(G)$, is the size of a maximum independent set.

Proposition

For every graph $G=(V,E)$, $\chi(G)\geq\omega(G)$ and $\chi(G)\geq|V|/\alpha(G)$.

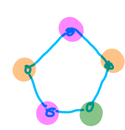
Proof

If there is a clique of size k, the k vertices in the clique must be of different colors.

For the second inequality, rewrite it as $\chi(G)\alpha(G)\geq|V|$. $\chi(G)$ is the number of color classes, and $\alpha(G)$ is the maximum size of a color class.

$\blacksquare$

The chromatic number is not necessarily the size of the maximal clique:



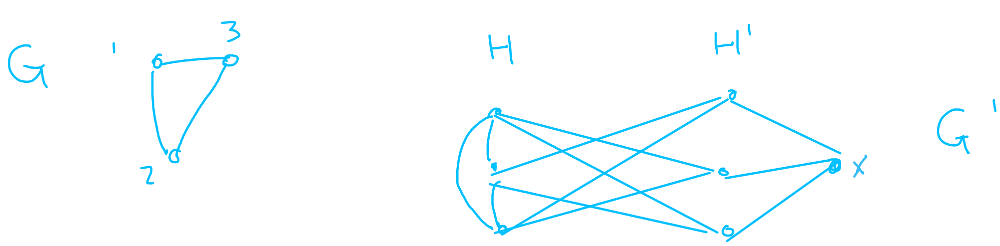
Maximal clique has size 2

Chromatic number is 3

Example: Mycielski's construction

From a simple graph G, construct a graph G' in the following way:

Let H and H' be two copies of G, but delete all edges from H'. If vertices u and v are adjacent in G, draw an edge between u in H and v' in H' (the copy of v in H'). Add an extra vertex x and connect it to all the vertices in H'.



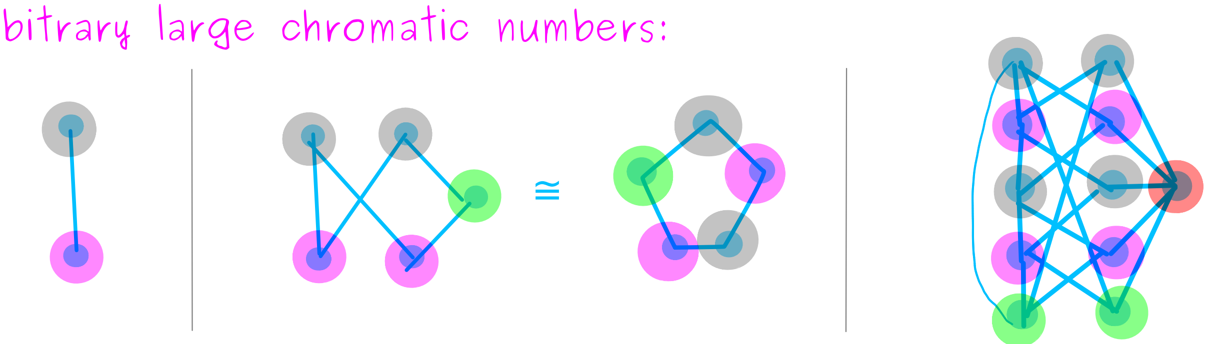
Notice that u and u' are never adjacent.

If G has chromatic number k, then G' has chromatic number $k+1$: The colors in H and in H' can be the same. In G, u and v can have the same color if they are not adjacent. Hence, u and v' (as well as v and u') are not adjacent in G', so they can have the same color. Hence x is the only vertex with a new color added.

So graphs obtained by iterating this process can have arbitraty large chromatic number.

Question: What is the clique number of a graph obtained with the Mycielski's contruction?

Mycielski's construction is used to build triangle-free graphs with arbitrary large chromatic numbers:

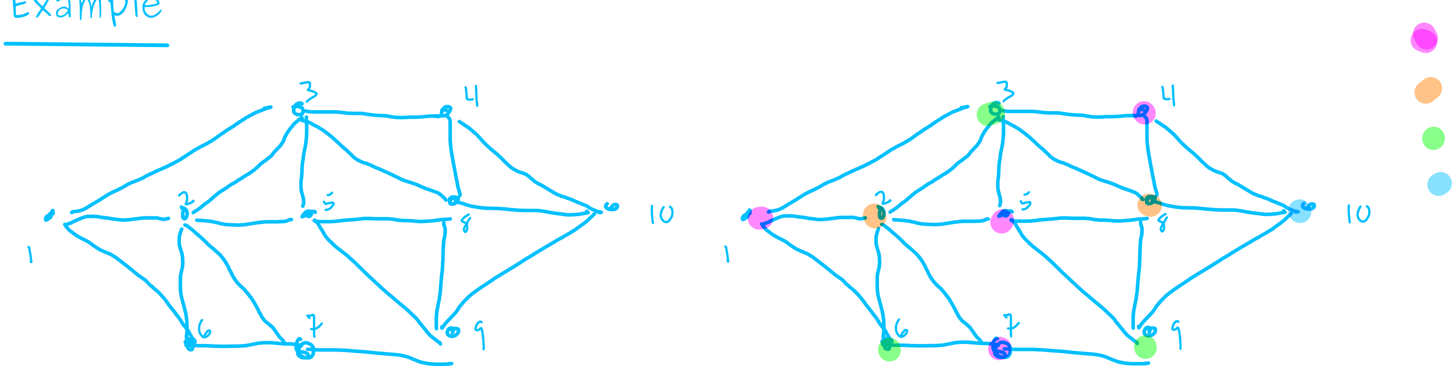


If we start from a triangle-free graph, the Mycielsky construction is triangle-free.

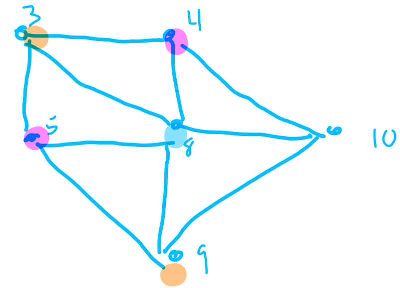
Greedy coloring algorithm

- Order the vertices $\{1,2,\dots,n\}$. We will color the vertices using numbers $\{1,2,\dots,n\}$.
- For every vertex (in order), label it with the smallest color not already in use in its neighborhood.

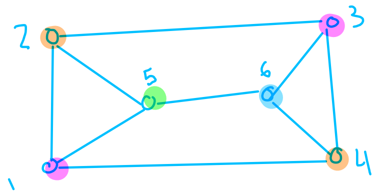
Example

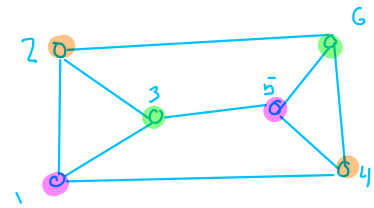


In this case, it is actually minimal. This graph cannot be colored with fewer than 4 colors.



The coloring does not always use the minimum number of colors:





Proposition

The chromatic number is at most $\Delta(G)+1$.

Proof

The greedy algorithm described above yields a proper coloring. In the worst case, all neighbors of one vertex have distinct color, and we must add a color. When this happens, the number of colors is one more than the number of neighbors; that is at most $\Delta(G)+1$.

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Reference: Douglas B. West. Introduction to graph theory, 2nd edition, 2001. Section 5.1.