Math 38 — Graph Theory

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# Bounds on the chromatic number

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Last class, I introduced proper colorings of graphs, and the chromatic number. We also looked at some bounds on the chromatic number, and we keep exploring bounds on the chromatic number today.

So far, we know:

- The chromatic number can be bounded in terms of the independence number and the clique number: $\chi(G)\geq\omega(G)$ and $\chi(G)\geq|V|/\alpha(G)$.
- The chromatic number can be bounded in terms of the maximum degree: $\chi(G)\leq\Delta(G)+1$.

These bounds are easy to check, but they are not the best possible.

## Another upper bound

**Theorem** (Brooks, 1941)
If $G$ is connected, and is not the complete graph nor an odd cycle, $\chi(G)\leq\Delta(G)$.

### Examples and special cases

If $\Delta(G)=0$,

If $\Delta(G)=1$,

If $\Delta(G)=2$,

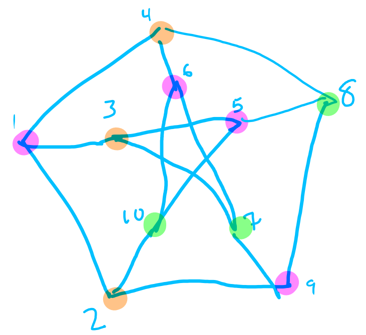
Complete graphs

The hypothesis that the graph is connected is needed to avoid the case of having only isolated vertices.


Not complete, maximum degree is $0$.
Chromatic number is $1$.

Notice that, whenever a graph with $n$ vertices is not the complete graph, the chromatic number is at most $n-1$: Since there is at least one pair of non-adjacent vertices in a non-complete graph, they can be the same colors. So $n$ colors are never needed if the graph is not complete.

Example: Coloring the Petersen graph using the greedy algorithm




The Petersen graph is 3-regular.
It satisfies the hypothesis of the theorem, so it must have maximum degree 3. That means there exists an ordering of the vertices that allows it.

Proof of Brooks' Theorem

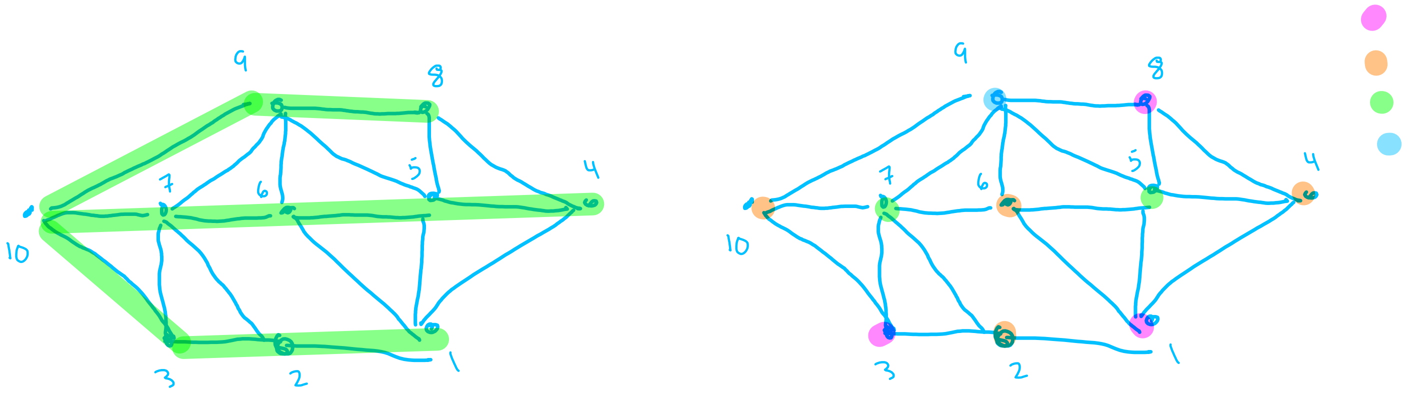
We already inspected the case where the largest degree is at most 2, so assume $\Delta(G)=k$ is at least 3.

If $G$ is not $k$-regular:
Then, there is a vertex $v$ with degree less than $k$. Let $T$ be a spanning tree in $G$ (which is possible since the graph is connected). We will use this spanning tree for ordering the vertices. The goal is to find the right ordering for the vertices, and then apply the greedy algorithm from last lecture.

- Number vertex $v$ with $n$ (last vertex to be colored).
- Label the other vertices in decreasing order on paths leaving $v$ in $T$.
- Color the vertices using the greedy algorithm from last lecture.

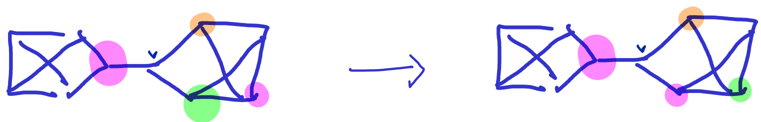
Every time we color a new vertex $u$ (that is not $v$), there are at most $k-1$ of its neighbors that have been previously colored, so $k$ colors are enough.

For the last step, we know that v has at most k−1 neighbors, so in the worst case, a k-th color will be necessary to color it.
In total, k colors are enough if the graph is not k-regular.

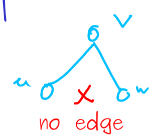


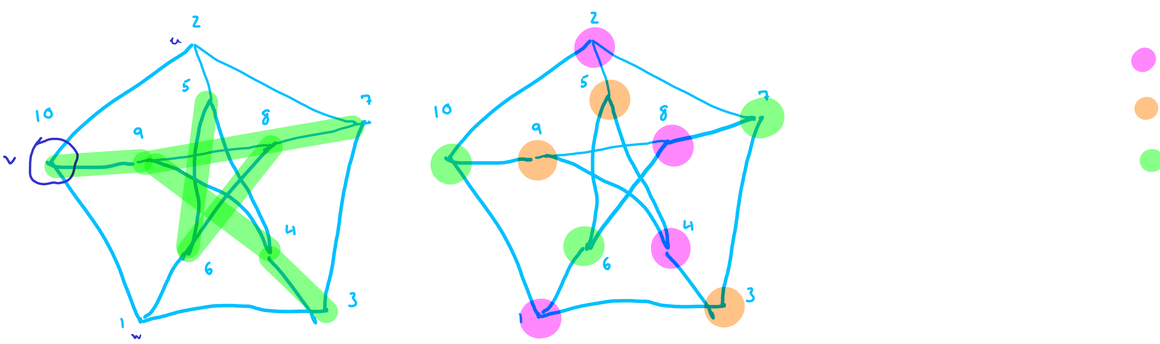

A similar process holds if the graph is k-regular, but there are two cases:

- There is a cut-vertex v. Then, G−{v} is disconnected, and each component can be colored with k colors. Place the colors in the components so that vertices incident to v have the same color in both components.
  Then, v can be colored using any other color, so G is k-colorable.



- There is no cut-vertex, meaning that G is 2-connected.
  If G has a vertex v with two neighbors that are not adjacent u and w such that G−{u,w} is connected, we can use a similar argument. We label u and w by 1 and 2, and create a spanning tree in G−{u,w}. Starting from v, we label the vertices in decreasing order and obtain a proper k-coloring of G because the last vertex has two vertices (u and w) colored the same.



I claim there is always such a triple of vertices when $G$ is 2-connected and $k$-regular, with $k \geq 3$. (The details of this are in the textbook.) ■

## Subgraph, cliques and chromatic number

**Proposition**

If $H$ is a subgraph of $G$, $\chi(H) \leq \chi(G)$.

**Proof**

■

This is similar to the proposition we had in last lecture: $\chi(G) \geq \omega(G)$. However, cliques are not needed to have large chromatic number (as exhibited by the graphs build using Mycielski's construction).

**Proposition**

Every $k$-chromatic graph has at least $\binom{k}{2}$ edges.

**Proof**

■

Reference: Douglas B. West. Introduction to graph theory, 2nd edition, 2001.
Sections 5.1 and 5.2