Math 38 — Graph Theory Nadia Lafrenière

Network and flows 5/6/2022

We progress in our journey to analyzing flow in a network. We first introduce line graphs (and digraphs) to express dual problems, and then move on to networks, flows and capacity.

Line graphs

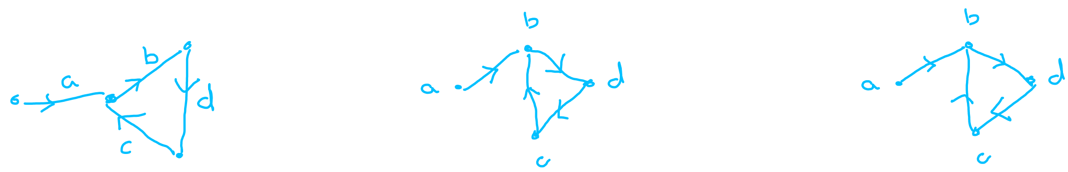
Goal: Introduce a way to translate edge Menger's theorem and other results on paths in terms of edges.

Let $G=(V,E)$ be a graph. Its line graph $L(G)$ has vertices E, and edges of $L(G)$ exist for two edges of G (vertices of $L(G)$) if they are incident to the same vertex in G.

Properties

- The number of edges in $L(G)$ is $\sum_{v\in V}\binom{d_G(v)}{2}$.
- In general, $G \not\cong L(G)$.
- For a graph with no isolated vertex, G is disconnected iff $L(G)$ is.

The same can be done with digraphs. In this case, there is a directed edge from e in E to f if there is a path in $D=(V,E)$ from e to f.



Theorem

If u and v are distinct vertices in a graph (or digraph) G, then the minimum size of an uv-disconnecting set (of edges) equals the maximum size of pairwise edge-disjoint uv-paths.

Sketch of the proof

Use Menger's theorem with $L(G)$.
Deleting an edge in G is equivalent to deleting a vertex in $L(G)$. So the minimum size of a uv-disconnecting set in G is the minimum size of a uv-cut in $L(G)$. By Menger's theorem, this is the maximum number of internally disjoint uv-paths in $L(G)$, which correspond to edge-disjoint paths in G.

Corollary

The edge-connectivity of a graph (or a digraph) is the maximum number k such that there is at least k edge-disjoint uv-paths for all pairs of vertices $\{u,v\}$.

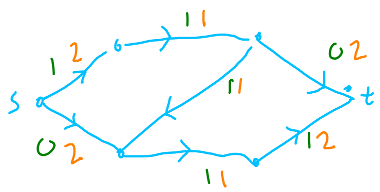
Maximum Network Flow

A network is a directed graph with a nonnegative capacity $c(e)$ on each edge e. A network has distinguished vertices: a source s and a sink t.

A flow f in a network assings a value $f(e)$ to edge e. For vertices, we write $f^+(v)$ for the total flow of the edges leaving v and $f^-(v)$ for the flow entering v.

A flow is feasible if
- $0 \le f(e) \le c(e)$ for every edge e. Capacity constraint
- $f^+(v) = f^-(v)$ for every vertex except source and sink
Conservation constraint

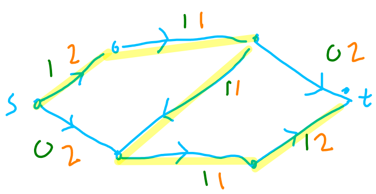
A feasible flow



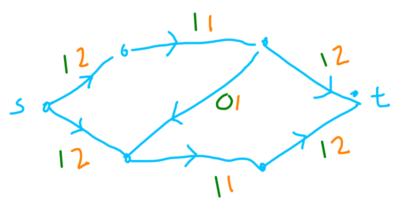
- capacity
- flow

The value of a flow is the net flow of the sink $(f^-(t) - f^+(t))$.
A maximum flow is a feasible flow of maximum value.

Maximal flow



Value 1



Value 2

Maximum flow

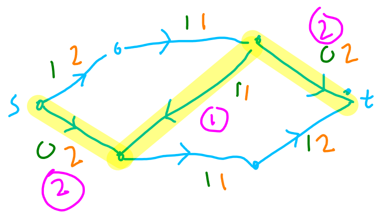
To increase the value of a maximal, but not maximum flow, we use f-augmenting paths. P is an <u>f-augmenting path</u> if

- it is going from source to sink.
- when P follows e in the forward direction, $f(e)<c(e)$.

Let $\varepsilon(e)=c(e)-f(e)$.

- when P follows e in the backward direction, $f(e)>0$. Let $\varepsilon(e)=f(e)$.

The <u>tolerance</u> of P is the minimum value of $\varepsilon(e)$ over edges in P.



Tolerance 1

<u>Lemma</u>

If P is an f-augmenting path with tolerance z, then we can create a flow f' with value value(f)+z in the following way:

- if e not in P, $f'(e)=f(e)$
- if e is forward in P, $f'(e)=f(e)+z$
- if e is backward in P, $f'(e)=f(e)-z$.

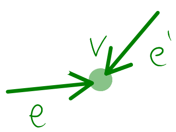
<u>Proof</u>

We must prove that f' is a flow (capacity and conservation constraints) and that the result has value z higher than the value of f.

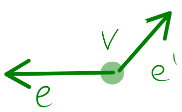
Capacity: If e is forward in P, then the tolerance of e was higher than z, so he can increase its flow by z.

If e is backward in P, then its flow is reduced by z, and it originally was higher than z, so it is still nonnegative.

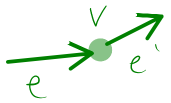
Conservation: if v is in P (but is neither s nor t), it has either two in-edges (one forward, one bacward), two out-edges also one each way), or one in- and one out-edge (in the same direction).



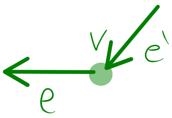
If the two edges are entering v, the flow for the e, the forward edge, is increased by z, and the flow for e' is decreased by z, so the entering flow is unchanged, and the conservation property is maintained.



The case of two out-edges is similar, as the exiting flow is decreased by z for the backward edge e, and increased by z for e'.



If the flow is forward for the two edges, one is entering and one is leaving, and they are both increased by z. Hence, $f'^{+}(v)=f^{+}(v)+z=f^{-}(v)+z=f'^{-}(v)$.



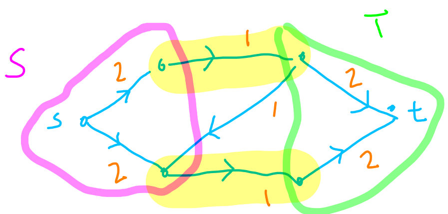
If the flow is backward for both edges, the proof is similar, as the flow is decreased on both edges.

In all cases, the conservation and capacity constraints are satisfied, so the flow is feasible.

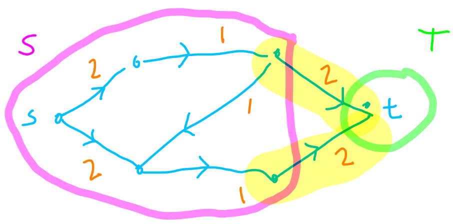
The flow is increased by z: P ends at the sink. So P is entering the sink, and $f'^{-}(t)=f^{-}(t)+z$ and $f'^{+}(t)=f^{+}(t)$ Therefore, $\text{value}(f')=\text{value}(f)+z$. ∎

Source/sink cut

Given a partition of the vertices in a network with source s and sink t, consider a partition of the vertices into a source set S (containing s) and a sink set T (with t). A source/sink cut is an edge cut $[S,T]$. Its capacity, $\text{cap}(S,T)$, is the total capacity of the edges from S to T.



Capacity 2



Capacity 4

Teaser for next class...

Theorem (Max-flow Min-cut, Ford-Fulkerson, 1956)
The maximum flow in a network is the minimum capacity of a source/sink cut.

Reference: Douglas B. West. Introduction to graph theory, 2nd edition, 2001. Sections 4.2, 4.3