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Network and flows

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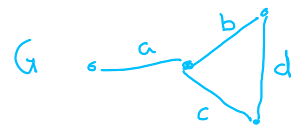
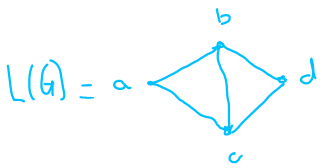
We progress in our journey to analyzing flow in a network. We first introduce line graphs (and digraphs) to express dual problems, and then move on to networks, flows and capacity.

Line graphs

Goal: Introduce a way to translate edge Menger's theorem and other results on paths in terms of edges.

Let $G=(V,E)$ be a graph. Its line graph $L(G)$ has vertices E, and edges of $L(G)$ exist for two edges of G (vertices of $L(G)$) if they are incident to the same vertex in G.





Properties

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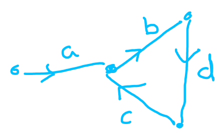
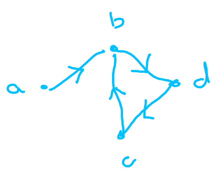
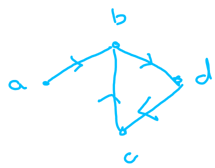
The same can be done with digraphs. In this case, there is a directed edge from e in E to f if there is a path in $D=(V,E)$ from e to f.







Theorem

If u and v are distinct vertices in a graph (or digraph) G, then the minimum size of an uv-disconnecting set (of edges) equals the maximum size of pairwise edge-disjoint uv-paths.

Sketch of the proof

Corollary

The edge-connectivity of a graph (or a digraph) is the maximum number k such that there is at least k edge-disjoint uv-paths for all pairs of vertices $\{u,v\}$.

Maximum Network Flow

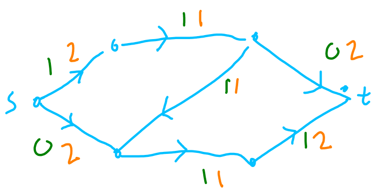
A network is a directed graph with a nonnegative capacity $c(e)$ on each edge e. A network has distinguished vertices: a source s and a sink t.

A flow f in a network assings a value $f(e)$ to edge e. For vertices, we write $f^+(v)$ for the total flow of the edges leaving v and $f^-(v)$ for the flow entering v.

A flow is feasible if
- $0 \le f(e) \le c(e)$ for every edge e. Capacity constraint
- $f^+(v) = f^-(v)$ for every vertex except source and sink

Conservation constraint

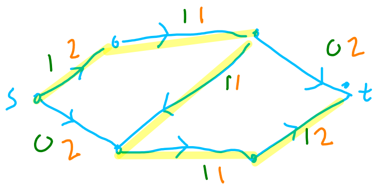
A feasible flow



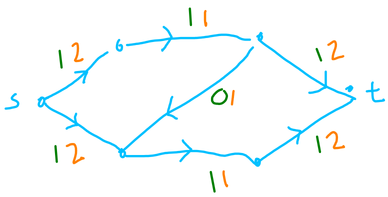
- capacity
- flow

The value of a flow is the net flow of the sink $(f^-(t) - f^+(t))$.
A maximum flow is a feasible flow of maximum value.

Maximal flow



Value 1



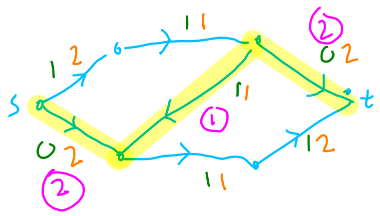
Value 2

Maximum flow

To increase the value of a maximal, but not maximum flow, we use f-augmenting paths. P is an f-augmenting path if
- it is going from source to sink.
- when P follows e in the forward direction, $f(e)<c(e)$.

Let $\varepsilon(e)=c(e)-f(e)$.
- when P follows e in the backward direction, $f(e)>0$. Let $\varepsilon(e)=f(e)$.

The tolerance of P is the minimum value of $\varepsilon(e)$ over edges in P.



Tolerance 1

Lemma

If P is an f-augmenting path with tolerance z, then we can create a flow f' with value value(f)+z in the following way:
- if e not in P, $f'(e)=f(e)$
- if e is forward in P, $f'(e)=f(e)+z$
- if e is backward in P, $f'(e)=f(e)-z$.

Proof

We must prove that f' is a flow (capacity and conservation constraints) and that the result has value z higher than the value of f.

Capacity:

Conservation:

The flow is increased by z:

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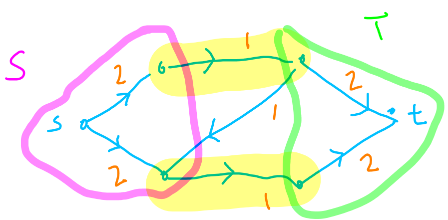
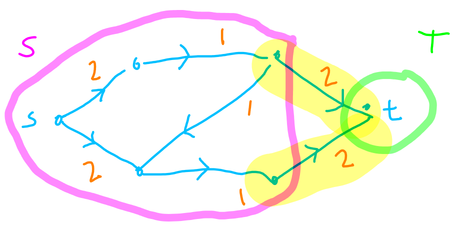
Source/sink cut

Given a partition of the vertices in a network with source s and sink t, consider a partition of the vertices into a source set S (containing s) and a sink set T (with t). A source/sink cut is an edge cut [S,T]. Its capacity, cap(S,T), is the total capacity of the edges from S to T.



Capacity 2



Capacity 4

Teaser for next class...

Theorem (Max-flow Min-cut, Ford-Fulkerson, 1956)
The maximum flow in a network is the minimum capacity of a source/sink cut.

Reference: Douglas B. West. Introduction to graph theory, 2nd edition, 2001. Sections 4.2, 4.3