Math 38 — Graph Theory
Connectivity and paths

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We keep looking at the interconnections between edge-connectivity and vertex-connectivity. We also consider what it means for cycles and paths.

Blocks

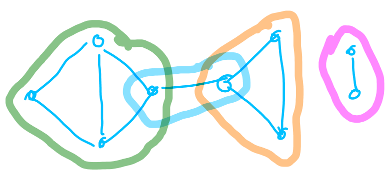
Is a connected graph with no cut-vertex 2-connected?


Connectivity 0


Connectivity 1

Definition

A block of a graph G is a maximal connected subgraph that has no cut-vertex.



Properties

- Isolated vertices, as well as "isolated edges" (isolated copies of K_2) are blocks.
- A cycle is always 2-connected, so it is always inside the same block.
- Since the only edges that are not in cycles are cut-edges, an edge with its two enpoints is a block if and only if it is a cut-edge.
- Blocks in a tree are edges (along with their two endpoints).
- Blocks in a loopless graph are its isolated vertices, its cut-edges and its 2-connected components.

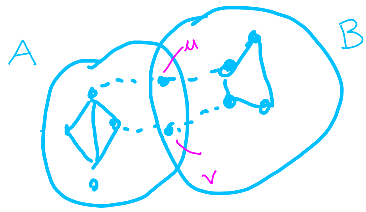
Proposition

Two blocks in a graph share at most one vertex.

Proof

By contradiction. If two blocks A and B share vertices u and v, they are connected components with no cut-vertices inside. They are also maximal, so if we extend their size, we will be creating a cut-vertex.

Since there is a path from u to v in A and one in B (because blocks are connected), there is a cycle containing u and v, and A and B form together a 2-connected component. Hence, they are in the same block. ∎

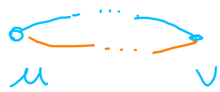


Proposition

If two blocks share a vertex, it is a cut-vertex.

2-connected graphs

Two paths from u to v are internally disjoint if they have no common internal vertex.



Theorem (Whitney, 1932)

A graph with at least three vertices is 2-connected if and only if there exist internally disjoint u,v-paths for each pair $\{u,v\}$.

Proof

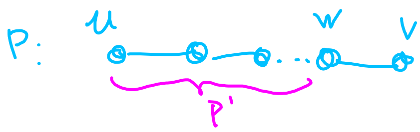
$\Leftarrow$ Since there are at least 2 disjoint u,v-paths for every pair $\{u,v\}$, u and v cannot be separated by removing one vertex. This is true for all $\{u,v\}$, so the graph does not have connectivity 1. It must have connectivity at least 2, and is hence 2-connected.

$\Rightarrow$ By induction on $d(u,v)$, the distance between u and v.

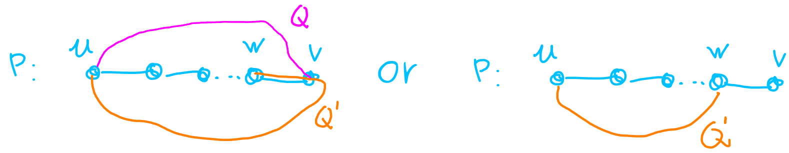
Base case: u and v are adjacent. Since the graph is 2-connected, it is also 2-edge-connected, and removing edge $e=\{u,v\}$ lets the graph connected, which means there is a path between u and v avoiding e.

Induction hypothesis: If distance is $k=d(u,v)$, there exists two internally disjoint uv-paths.

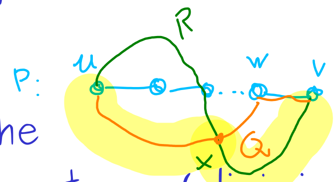
Induction step: Let u and v be at distance $k+1$, and let P be a uv-path of (minimal) length $k+1$. Let w be the vertex on P at distance k of u, so w is adjacent to v, and P' be that portion of P.



By induction hypothesis, there exist two internally disjoint uw-paths, P' and Q'.
If Q' contains vertex v, let Q be the portion from u to v in Q'; then Q is a uv-path that is internally disjoint from P.



Otherwise, consider $G-w$. It is connected since there is no cut-vertex. So there is a path R between u and v avoiding w. If it avoids P or Q, R is internally disjoint from it. Otherwise, let x be the last vertex of R that also belongs to either P or Q. If x belongs to Q, then P is disjoint from the part of Q between u and x and from the part of R between x and v, which is a path from u to v (disjoint from P). If x belongs to P, the argument is similar.



Corollary

For a graph with at least three vertices, the following conditions are characterization of 2-connected graphs:

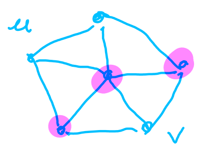
(A) G is connected and has no cut-vertex.

(B) For every pair of vertices $\{u,v\}$, there are internally disjoint u,v-paths.

(C) For every pair of vertices $\{u,v\}$, there is a cycle through u and v.

Menger's theorem

Given two vertices u and v, a uv-cut is a set of vertices S such that $G-S$ has no uv-path.



Let $\kappa(u,v)$ be the size of a minimum uv-cut.

Proposition

For u and v vertices of G, $\kappa(u,v) \geq \kappa(G)$.

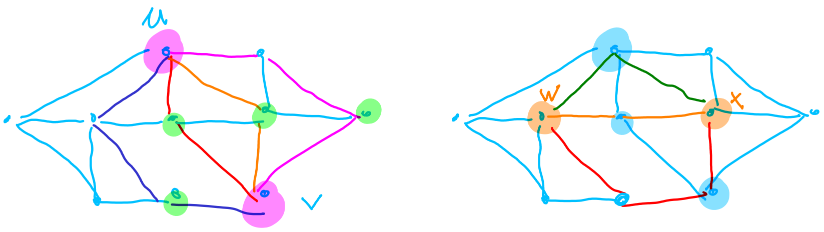
Proof

~~A uv~~-cut makes the graph disconnected, so the connectivity is at most the size of a uv-cut.

Let $\lambda(u,v)$ be the maximum number of internally disjoint uv-paths.

Proposition
For u and v vertices of G, $\kappa(u,v) \geq \lambda(u,v)$.

Proof
We need to delete at least one vertex per path, and no vertex belongs to two paths.



Minimal uv-cut, size 4
Minimal wx-cut, size 3

In fact, one can get a much stronger result:

Theorem (Menger, 1927)
If u and v are not adjacent, the minimum size of a uv-cut is the maximum number of internally disjoint uv-paths.

Proof (optional): read in the textbook, proof of theorem 4.2.17.
We will see another proof with the Ford-Fulkerson algorithm next week Monday.

Reference: Douglas B. West. Introduction to graph theory, 2nd edition, 2001. Sections 4.1 and 4.2