Math 38 - Spring 2022 4/28/2022

Solutions

#1 a) • Leaves are vertices of degree 1

• In a tree, $|E| = |V|-1$

• In any graph, $\sum_{v \in V} d(v) = 2|E|$.

Then,

$$2|V|-2 = \sum_{v \in V} d(v) = 2+3+4 + \#\{\text{leaves}\}$$

and

$$2|V|-2 = 6-2+2\#\{\text{leaves}\}$$

Then, $\#\{\text{leaves}\} = 5$.

b) In general, the number of trees with degree sequence $(d_1^{k_1}, d_2^{k_2}, \ldots, d_{n-1}^{k_{n-1}})$ is $\binom{|V|-2}{k_1-1, k_2-1, \ldots, k_{n-1}-1}$,

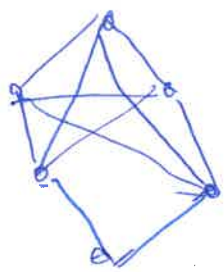
where $d_i^{k_i}$ means that d_i appears k_i times in the degree sequence.

Then, $\binom{6}{3,2,1} = \frac{6!}{3!\,2!\,1!} = \frac{6 \cdot 5 \cdot 4}{2} = 60$, and there

are 60 labeled trees with vertex 1 of degree 4, vertex 2 of degree 3, and vertex 3 of degree 2, and leaves. We also need to account for which vertices go in the sequence: there are $8 \cdot 7 \cdot 6$ such choices. Hence, the number of trees is

$8 \cdot 7 \cdot 6 \cdot 60 = 20{,}160$ trees.

#2 This is false. Here is a counter-example.



- Eulerian
- Simple
- 6 vertices
- 11 edges

#3 The maximum number of edges in a bipartite subgraph of K_n is $\left\lfloor \frac{n^2}{4} \right\rfloor$.

Two options for the proof:

1- We saw in class that triangle-free graphs have at most $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges. Bipartite graphs are triangle-free.

Also, the example we saw is bipartite, since it is

$K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$.

2- The example is the same.

Also, if G is bipartite, there are independent sets T and S that form a partition of V.

Each edge is between a vertex of T and a vertex of S, so the maximum number of edges is $|T| \cdot |S|$. Also, $|T| + |S| = n$.

To maximize $|T| \cdot |S| = |T| (n - |T|)$, we use optimization to get $|T| = \frac{n}{2}$. Then, the maximum number of edges occur for the complete bipartite graph with vertices distributed as equally as possible between the sets.

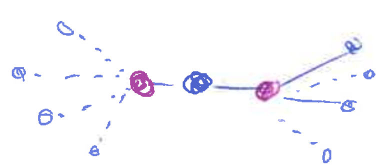
4

(a) Spanning trees in $K_{2,m}$.

- There must be exactly one vertex that is adjacent in the tree to the 2 vertices in the first independent set. (m options)
- For the m-1 remaining vertices, choose each of the 2 vertices in the other set to connect it to. 2 options, m-1 times
- Each choice is independent, so we can use the product principle.

In total, $\tau(K_{2,m}) = m2^{m-1}$.

(b) We noticed in part (a) that every spanning tree will look like



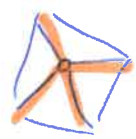
m blue vertices.

Therefore, to count trees up to isomorphism, we need to count the number of possible sets of size for the two subsets of m-1 vertices. Those are $\{0, m-1\}, \{1, m-2\}, \ldots \{\lfloor \frac{m-1}{2} \rfloor, \lceil \frac{m-1}{2} \rceil\}$.

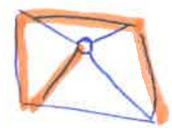
Hence, there are $\left\lceil \frac{m+1}{2} \right\rceil$ isomorphism classes of spanning trees.

#5 (a) True. this is the number of edges, which is $n-1$, with n the order of G.

(b) False. Counter-example.

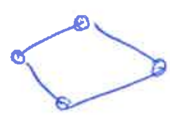


diameter 2.



diameter 4

#6. False. The parity of the cycle does not matter.

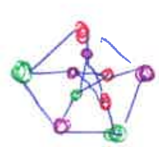


No odd-cycle
No cut-edge.

Hence, we prove that the statement $\Leftarrow$ is wrong.

#7 True, because the sum of the degrees in each component must be even.

#8.
- It has 15 edges and 10 vertices
- It is simple.
- It can be partitioned into three independent sets



- It is 3-regular
- It is not Eulerian
- It has diameter 2.

- It can be decomposed into paths of length 3, E-graph, H-graph, and T-graphs, as well as edges.

- Its vertices can represent subsets of $\{1,2,3,4,5\}$ of size 2.
- It is a great counter-example to many statements
- It is triangle-free.
- Its girth is 5
- It has 2000 spanning trees (you can say only that it has at least one).
- It is connected.

#9 I really like watching you all solving graph theory problems :)