Math 38 — Graph Theory

Nadia Lafrenière

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Connectivity

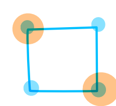
Cuts and connectivity

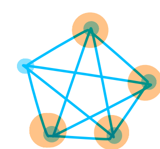
A vertex cut (or separating set) is a subset of vertices S such that $G-S$ has more than one component.

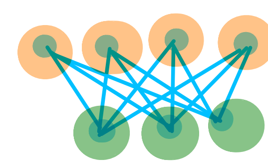
The connectivity of G, $\kappa(G)$, is the minimum size of a separating set, if it exists, or $n-1$.

A graph is k-connected if its connectivity is at least k.

Examples





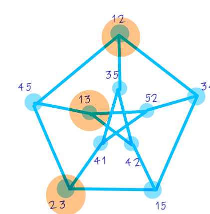


Disconnected = connectivity 0

Connected = 1-connected

Cycles of length at least 3 have connectivity 2

Petersen graph has connectivity 3.



Complete graph K_n has connectivity $n-1$.

Complete bipartite graph $K_{m,n}$ has connectivity $\min\{n,m\}$.

By convention, we say the graph with one vertex has connectivity 0.

Proposition

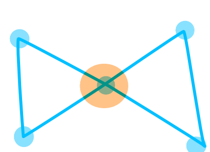
The connectivity of a connected graph is at most its minimum degree.

Proof

One can isolate a single vertex by removing all the vertices around it.

Remark

The connectivity of a connected graph is not at least its minimum degree.

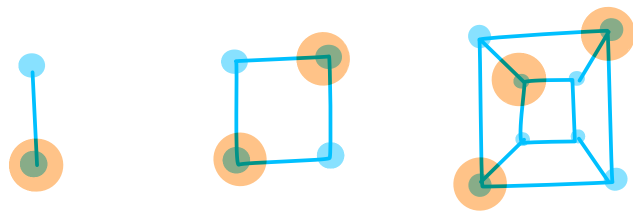


Minimum degree 2, but there is a cut-vertex => connectivity 1.

Example

The hypercube H_k has connectivity k.

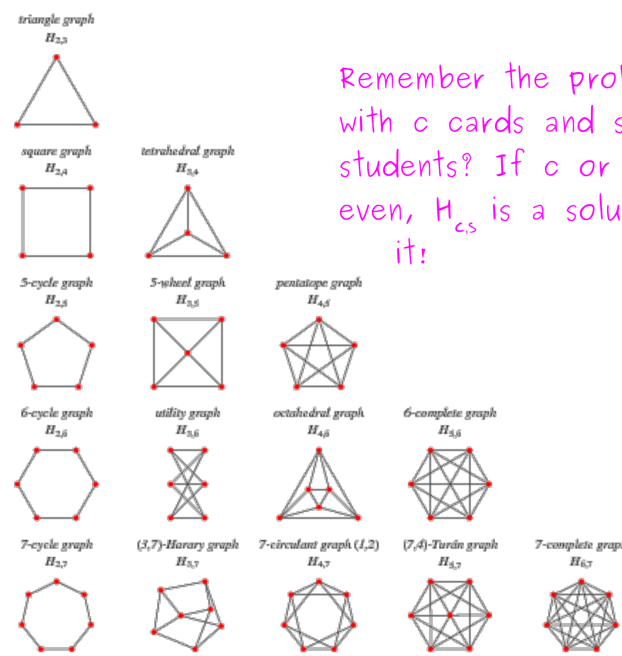
Of course, since it is k-regular, it has connectivity at most k.
We can prove by induction it has connectivity at least k:



Example: Harary graphs

Harary graphs $H_{k,n}$ are graphs with n vertices and $\lceil \frac{nk}{2} \rceil$ edges, $2 \le k < n$, being as regular as possible.
They have connectivity k:
- k is the minimum degree of $H_{n,k}$

There is a proof in the textbook that it has connectivity at least k.



Remember the problem with c cards and s students? If c or s is even, $H_{c,s}$ is a solution to it!

Theorem (Harary, 1962)

Let $k>2$. The minimum number of edges in a k-connected graph with n vertices is $\lceil \frac{nk}{2} \rceil$.

Proof

This is an example of an extremal problem:

- There cannot be fewer edges in a k-connected graph. Since G is k-connected, the minimum degree is at most k. Then, there must be at least $\lceil \frac{nk}{2} \rceil$ edges.

- Example of k-connected graphs with n vertices and $\lceil \frac{nk}{2} \rceil$ edges are the Harary graphs.

Edge-connectivity

What if we instead consider the number of edges we need to remove to disconnect a graph?

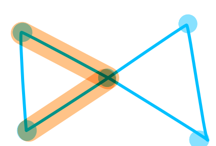
Definition

A disconnecting set is a subset of edges $F \subseteq E$ such that $G-F$ has at least 2 components.

separating $\neq$ disconnecting

The edge-connectivity is the minimum size of a disconnecting set, and is noted $\kappa'(G)$. A graph is k-edge-connected if it has edge-connectivity at least k.

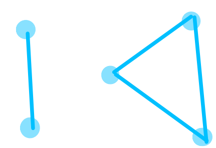
Examples

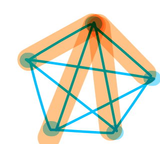


$\mapsto$

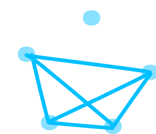
$\kappa(a)=1$

$\kappa'(a)=2$





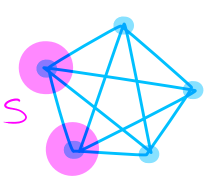
$\mapsto$



$\kappa(a)=\kappa'(a)=4$

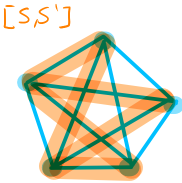
Complete graphs have edge-connectivity $n-1$. You can prove it!

Let $S \subseteq V$ be a vertex subset of a connected graph G. Let $[S,\bar{S}]$ be the set of all edges with one endpoint in S and one in $\bar{S}$. Then $[S,\bar{S}]$ is an edge cut.

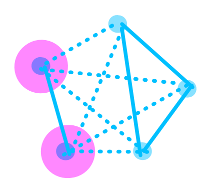


$\mapsto$

$[S,S']$

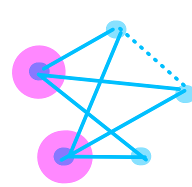


$\mapsto$



Edge cut $\underset{\not\Leftarrow}{\Rightarrow}$ Disconnecting set

Not an edge cut $\rightarrow$



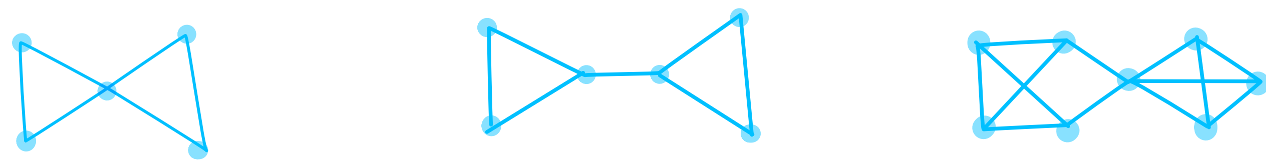
Edge cut $\Leftrightarrow$ Minimal disconnecting set

Connection to vertex-connectivity

Theorem (Whitney, 1932)
If G is simple, then $\kappa(G) \leq \kappa'(G) \leq \delta(G)$. In words: vertex-connectivity is at most edge-connectivity, which is always at most the smallest degree.

Example of inequalities

$\kappa(G) < \kappa'(G) = \delta(G)$ $\qquad$ $\kappa(G) = \kappa'(G) < \delta(G)$ $\qquad$ $\kappa(G) < \kappa'(G) < \delta(G)$



Proof
We first prove $\kappa'(G) \leq \delta(G)$. Let v be a vertex with degree $\delta(G)$. The edge cut for the set $\{v\}$ has $\delta(G)$ edges, so an edge cut with $\delta(G)$ edges exist, and the minimum edge cut has size at most $\delta(G)$.

We also need to prove $\kappa(G) \leq \kappa'(G)$. To do so, we start with a minimum edge cut, and construct a vertex cut with at most the same size. If this process is always possible, that proves the desired inequality.

Consider a minimum edge cut $[S, V-S]$. There are two cases:

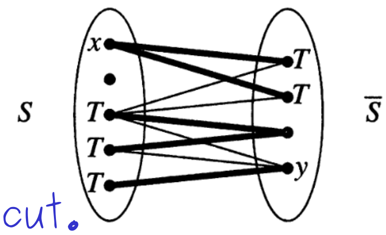
- If every vertex of S is connected to every vertex of $V-S$, then $\#[S, V-S] = |S||V-S| \geq |V|-1$. Also, by definition, $\kappa(G) \leq |V|-1$. So $\kappa(G) \leq |V|-1 \leq \#[S, V-S] = \kappa'(G)$ (the last equality is because the minimum edge-cut is the minimum disconnecting set.
- Otherwise, there is one vertex x in S and y not in S that are not adjacent. We construct a set of vertices T:

    - All neighbors of x in $V-S$.
    - All vertices of $S \setminus \{x\}$ that are adjacent to vertices in $V-S$.

Then, T is a vertex cut: There is no way to go from x to y without passing through one edge of T, so $G-T$ is disconnected. We need to show that T has at most $\#[S, V-S]$ vertices.

For each vertex t of T:

- If t is a neighbor of x, then xt is in the edge cut.
- If t is in S, then t is adjacent to at least one vertex u in $V-S$. Then ut is in the edge cut.



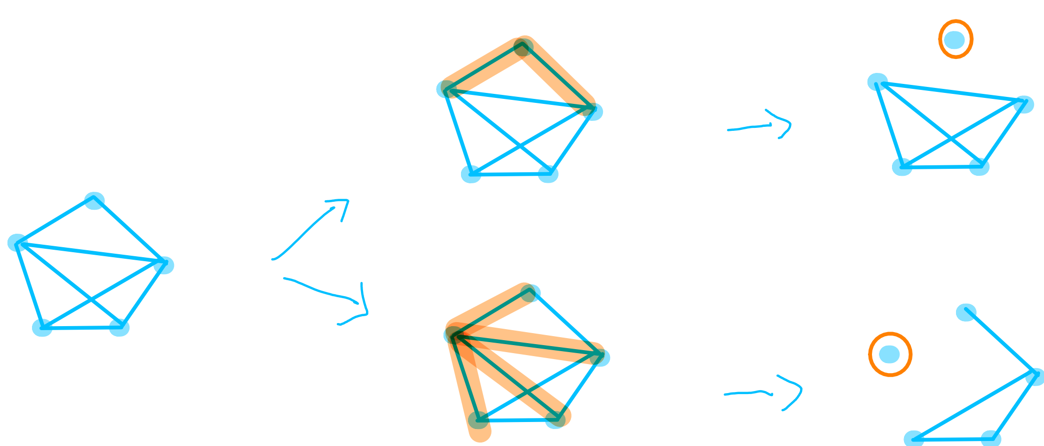
No edge is counted twice in this list, because x is not in T.
Since every edge in this list is in the edge cut, then $|T| \leq \#[S, V-S]$, and $\kappa(G) \leq \kappa'(G)$.

Proposition

Let G be a connected graph. Then, an edge cut F is minimal if and only if $G-F$ has exactly two components.

Remark

If we replace minimal by minimum, then the statement becomes false: $G-F$ can have two components while there are edge cuts with size smaller than $|F|$.



With your study group, try to agree on an explanation of why this is true.

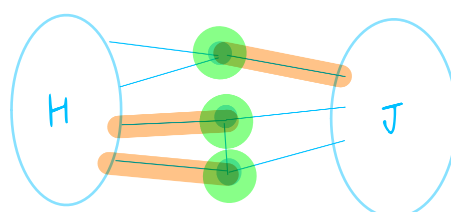
Edge connectivity for regular graphs

Theorem

If G is a 3-regular graph, then $\kappa(G)=\kappa'(G)$.

Proof

We already know that $\kappa(G)\leq\kappa'(G)$, in general. To prove the statement, we only need to show the reverse inequality ($\geq$), that is, from a minimum vertex cut, create an edge cut of the same size.
Let S be a minimum vertex cut, and let H and J be two components of $G-S$. Since S is minimum, every vertex of it has a neighbor in H and a neighbor in J. Also a vertex cannot have at least two neighbors in both H and J since G is 3-regular. For each vertex v in S, delete the edge from v to the component in which it has only one neighbor (if there is one neighbor in H, one in J and another one (in S for example), delete the edge to H).



That process breaks all the paths between H and J, so the deleted edges form an edge cut. Also, the size of that edge cut is $|S|$, which proves the statement. ∎

Reference: Douglas B. West. Introduction to graph theory, 2nd edition, 2001. Section 4.1