Cuts and connectivity

A vertex cut (or separating set) is a subset of vertices S such that
G−S has more than one component.

The connectivity of G, κ(G), is the minimum size of a separating set,
if it exists, or n−1.
A graph is k-connected if its connectivity is at least k.
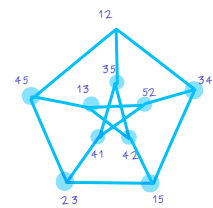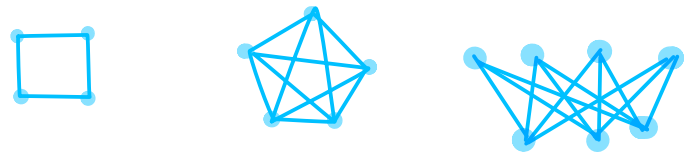
Examples

Disconnected =
Connected =
Cycles of length at least 3 have
Petersen graph has
Complete graph K_n has
Complete bipartite graph $K_{m,n}$ has
By convention, we say the graph with one vertex has connectivity 0.

Proposition
The connectivity of a connected graph is at most its minimum degree.

Proof
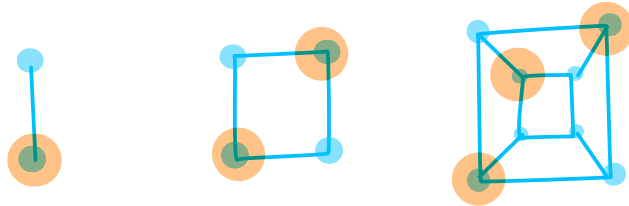One can isolate a single vertex by removing all the vertices around it.

Remark
The connectivity of a connected graph is not at least its minimum
degree.

Minimum degree 2, but there is a
cut-vertex => connectivity 1.

Example
The hypercube H_k has connectivity k.

Of course, since it is k-regular, it has connectivity at most k.
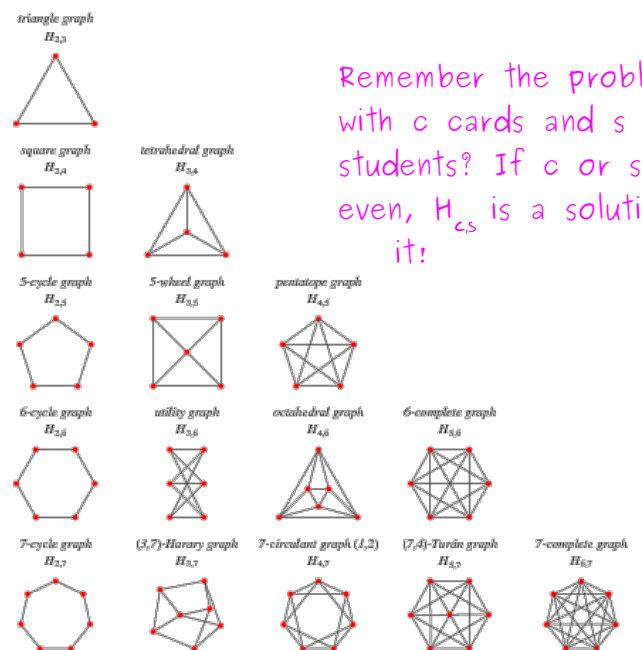We can prove by induction it has connectivity at least k:

Example: Harary graphs
Harary graphs $H_{k,n}$ are graphs with n vertices and $\lceil \frac{nk}{2} \rceil$ edges, $2 \le k < n$, being as regular as possible.
They have connectivity k:
— k is the minimum degree of $H_{n,k}$

There is a proof in the textbook that it has connectivity at least k.

triangle graph $H_{2,3}$

square graph $H_{2,4}$ *tetrahedral graph* $H_{3,4}$

5-cycle graph $H_{2,5}$ *5-wheel graph* $H_{3,5}$ *pentatope graph* $H_{4,5}$

6-cycle graph $H_{2,6}$ *utility graph* $H_{3,6}$ *octahedral graph* $H_{4,6}$ *6-complete graph* $H_{5,6}$

7-cycle graph $H_{2,7}$ *(3,7)-Harary graph* $H_{3,7}$ *7-circulant graph (1,2)* $H_{4,7}$ *(7,4)-Turán graph* $H_{5,7}$ *7-complete graph* $H_{6,7}$

Remember the problem with c cards and s students? If c or s is even, $H_{c,s}$ is a solution to it!

Theorem (Harary, 1962)
Let $k > 2$. The minimum number of edges in a k-connected graph with n vertices is $\lceil \frac{nk}{2} \rceil$.

Proof

What if we instead consider the number of edges we need to remove to disconnect a graph?

Definition

A <u>disconnecting set</u> is a subset of edges $F \subseteq E$ such that $G-F$ has at least 2 components.

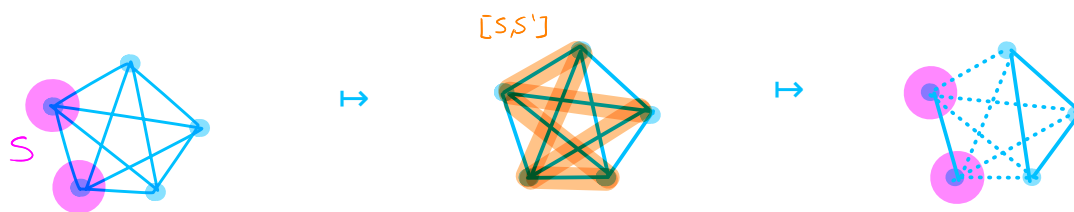separating ≠ disconnecting

The <u>edge-connectivity</u> is the minimum size of a disconnecting set, and is noted $\kappa'(G)$. A graph is <u>k-edge-connected</u> if it has edge-connectivity at least k.
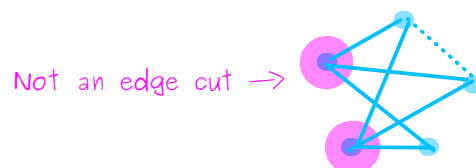
Examples

↦

$\kappa(a)=1$
$\kappa'(a)=2$

↦

$\kappa(a)=\kappa'(a)=4$

Complete graphs have edge-connectivity $n-1$. You can prove it!

Let $S \subseteq V$ be a vertex subset of a connected graph G. Let $[S,\bar{S}]$ be the set of all edges with one endpoint in S and one in $\bar{S}$. Then $[S,\bar{S}]$ is an <u>edge cut</u>.

$[S,S']$

↦   ↦

S

Edge cut ⇄̸ Disconnecting set

Not an edge cut →

Edge cut ⇔ Minimal disconnecting set

Connection to vertex-connectivity

Theorem (Whitney, 1932)

If G is simple, then $\kappa(G) \le \kappa'(G) \le \delta(G)$. In words: vertex-connectivity is at most edge-connectivity, which is always at most the smallest degree.

Example of inequalities

$$\kappa(G) < \kappa'(G) = \delta(G) \qquad \kappa(G) = \kappa'(G) < \delta(G) \qquad \kappa(G) < \kappa'(G) < \delta(G)$$

Proof

We first prove $\kappa'(G) \le \delta(G)$. Let v be a vertex with degree $\delta(G)$. The edge cut for the set $\{v\}$ has $\delta(G)$ edges, so an edge cut with $\delta(G)$ edges exist, and the minimum edge cut has size at most $\delta(G)$.

We also need to prove $\kappa(G) \le \kappa'(G)$. To do so, we start with a minimum edge cut, and construct a vertex cut with at most the same size. If this process is always possible, that proves the desired inequality.

Consider a minimum edge cut $[S, V-S]$. There are two cases:
- If every vertex of S is connected to every vertex of $V-S$, then $\#[S, V-S] = |S||V-S| \ge |V|-1$. Also, by definition, $\kappa(G) \le |V|-1$. So $\kappa(G) \le |V|-1 \le \#[S, V-S] = \kappa'(G)$ (the last equality is because the minimum edge-cut is the minimum disconnecting set.
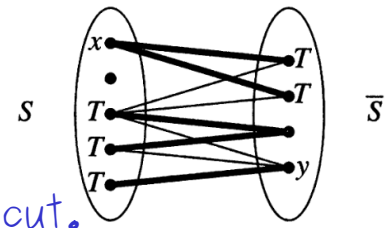- Otherwise, there is one vertex x in S and y not in S that are not adjacent. We construct a set of vertices T:

    - All neighbors of x in $V-S$.
    - All vertices of $S \setminus \{x\}$ that are adjacent to vertices in $V-S$.

Then, T is a vertex cut: There is no way to go from x to y without passing through one edge of T, so G−T is disconnected. We need to show that T has at most #[S, V−S] vertices.

For each vertex t of T:
− If t is a neighbor of x, then xt is in the edge cut.
− If t is in S, then t is adjacent to at least one vertex u in V−S. Then ut is in the edge cut.

No edge is counted twice in this list, because x is not in T. Since every edge in this list is in the edge cut, then $|T| \leq$ #[S, V−S], and $\kappa(G) \leq \kappa'(G)$.

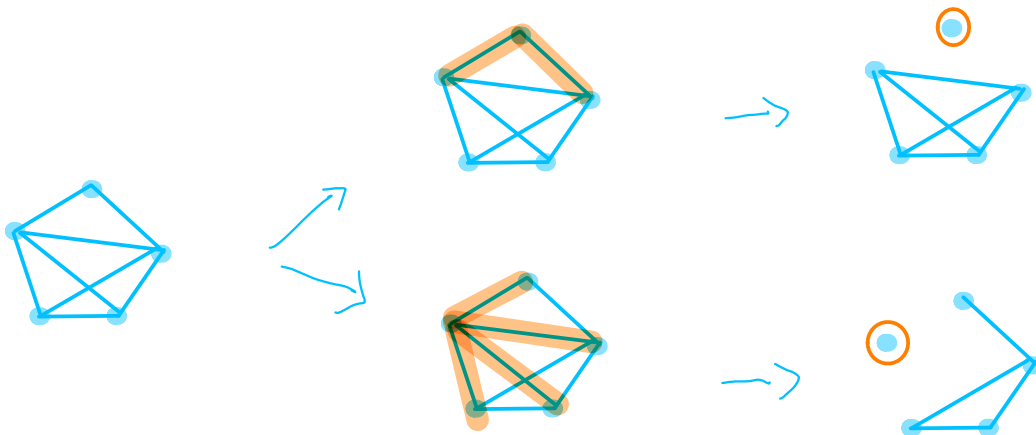## Proposition
Let G be a connected graph. Then, an edge cut F is minimal if and only if G−F has exactly two components.

Remark
If we replace minimal by minimum, then the statement becomes false: G−F can have two components while there are edge cuts with size smaller than |F|.

With your study group, try to agree on an explanation of why this is true.
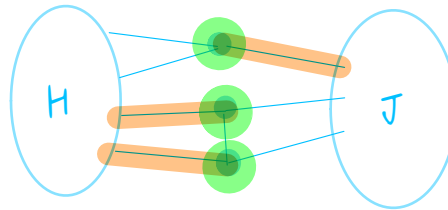
Edge connectivity for regular graphs

Theorem
If G is a 3-regular graph, then $\kappa(G) = \kappa'(G)$.

Proof
We already know that $\kappa(G) \leq \kappa'(G)$, in general. To prove the statement, we only need to show the reverse inequality ($\geq$), that is, from a minimum vertex cut, create an edge cut of the same size.

Let S be a minimum vertex cut, and let H and J be two components of G−S. Since S is minimum, every vertex of it has a neighbor in H and a neighbor in J. Also a vertex cannot have at least two neighbors in both H and J since G is 3-regular. For each vertex v in S, delete the edge from v to the component in which it has only one neighbor (if there is one neighbor in H, one in J and another one (in S for example), delete the edge to H).

That process breaks all the paths between H and J, so the deleted edges form an edge cut. Also, the size of that edge cut is |S|, which proves the statement.