Math 38 — Graph Theory

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# Covers, matchings and independent sets: Min-max theorems

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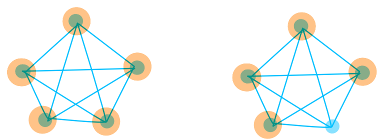
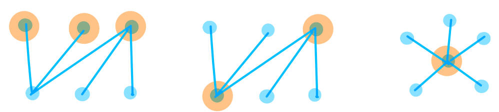
Recall from last class that a matching is a subset of edges such that no vertex appears twice as endpoints. We compare these notions with those of edge covers and vertex covers.

A <u>vertex cover</u> is a set S of vertices of G that contains at least one endpoint of every edge of G. The vertices in S cover G.

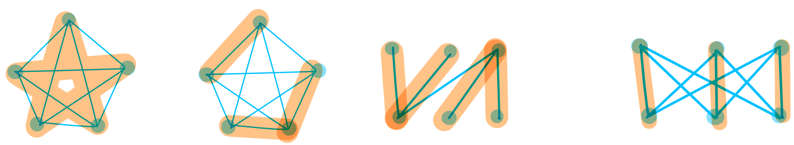
<u>Examples</u>



A cover of $K_n$ has at least size $n-1$.



An <u>edge cover</u> is a set of edges of G that contains as endpoints every vertex of G.



The minimum number of edges in an edge cover is $\#V/2$.

| Sets | Items | Property |
|---|---|---|
| Matchings | Edges | At most one edge per vertex |
| Edge covers | Edges | At least one edge per vertex |
| Vertex covers | Vertices | At least one vertex per edge |
| Independent set | Vertices | At most one vertex per edge |

When are:

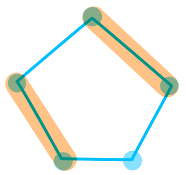
- Vertex covers and independent sets equal?
- Matchings and edge covers equal?

# Matchings and vertex covers

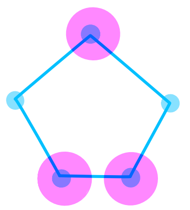
From the table above, it might seem that matchings and vertex covers are not related. However, consider a matching. In a vertex cover, every edge of the matching has to be covered by one of its endpoint. So the edge uv has either u or v (or both) in the vertex cover. Also, u (and v) cannot belong to more than one edge in the matching. So we have the following proposition:

**Proposition**

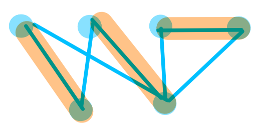
If M is a matching of G and S is a vertex cover of the same graph G, $|M| \leq |S|$.



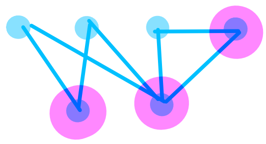
Largest matching: size 2.



Smallest vertex cover: size 3.



Largest matching: size 3.



Smallest vertex cover: size 3.

In the last example, the matching and the vertex cover have the same size ($|M| = |S|$). Of course, one can get smaller matchings and larger vertex covers, but the first example shows it is not possible to get a matching and a vertex cover of the same size.

**Remark**

The statements

"For any matching M and any vertex cover S, $|M| \leq |S|$"

and

"The maximum size of a matching is always at most the minimum size of a cover"

are equivalent.

## Theorem (König, Egerváry, 1931, independently)

If $G$ is a bipartite graph, then the maximum size of a matching in $G$ is equal to the minimum size of a vertex cover in $G$.

## Proof

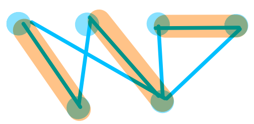
Available as, either:

- the proof of Theorem 3.1.16 in the textbook.
- the short proof of Romeo Rizzi (the paper is on Canvas).

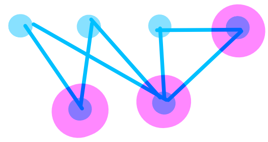
The first one is a proof by construction, the second one is a proof by contradiction.

## Remark

This is not an if and only if statement. For example, the graph below contains a triangle, but has a matching a vertex cover of the same size.

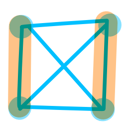


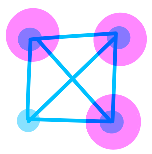
Largest matching: size 3.



Smallest vertex cover: size 3.

Also, being a perfect matching is not enough, as shown by this example with just 4 vertices.





Complete graphs have vertex covers of size $n-1$ and maximum matching of size $\lfloor \frac{n}{2} \rfloor$

## Remark

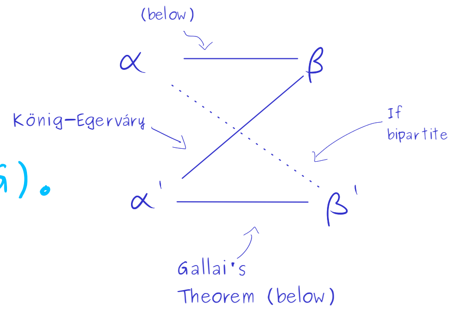
The theorem above is an example of a min-max relation: that means that solving an optimization problem calling for the minimum of something in a graph is equivalent to solving an optimization problem for the maximum. Here: the maximum matching and the minimum vertex cover. We will see more of them over the term.

# Optimization problems

| Sets | Items | Problem | Notation |
|---|---|---|---|
| Matchings | Edges | Finding maximum | $\alpha'(G)$ |
| Edge covers | Edges | Finding minimum | $\beta'(G)$ |
| Vertex covers | Vertices | Finding minimum | $\beta(G)$ |
| Independent set | Vertices | Finding maximum* | $\alpha(G)$ |

* This is defined as the <u>independence number</u>

As an example on how to use these notations, König–Egerváry theorem states that $\alpha'(G)=\beta(G)$ for bipartite graphs. For any graph $G$, $\alpha'(G)\leq\beta(G)$.




**Lemma** ($\alpha-\beta$)
In a graph $G=(V,E)$, $S$ is an independent set if and only if $V-S$ is a vertex cover. Hence, $|V|=\alpha+\beta$.

**Proof**
Let $S$ be an independent set, meaning there is no edge between two vertices of $S$; every edge has at least one endpoint in $V-S$. That means that $V-S$ is a vertex cover.
Conversely, if $S'$ covers $G$, every edge has at least one endpoint in $S'$, so $V-S'$ is independent. ∎

**Theorem** (Gallai, 1959, $\alpha'-\beta'$)
Let $G$ be a graph with $n$ vertices, none of them being isolated.
Then, $\alpha'(G)+\beta'(G)=n$. → size of maximum matching + minimum edge cover

**Sketch of proof**
2 steps:
1. From a maximum matching (of size $\alpha'(G)$), construct an edge cover of size $n-\alpha'(G)$. That implies that $n-\alpha'(G)\geq\beta'(G)$.
2. From a minimum edge cover (of size $\beta'(G)$), construct a matching of size $n-\beta'(G)$. That implies that $\alpha'(G)\geq n-\beta'(G)$.

These two steps prove that $\alpha'(G)+\beta'(G)=n$.

<u>Corollary</u> (König, 1916, $\alpha-\beta'$)
If $G$ is a bipartite graph with no isolated vertex, then the size of a maximum independent set is the size of a minimum edge cover.

Reference: Douglas B. West. Introduction to graph theory, 2nd edition, 2001. Section 3.1