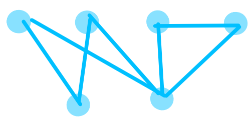
Math 38 — Graph Theory Nadia Lafrenière

Maximum and perfect matchings

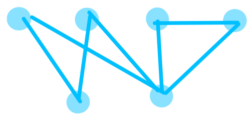
04/27/2022

In a graph, a <u>matching</u> is a subgraph with maximal degree 1 (so every vertex is connected to at most one other vertex).

A maximal, but not maximum matching



A maximum matching, also a perfect matching

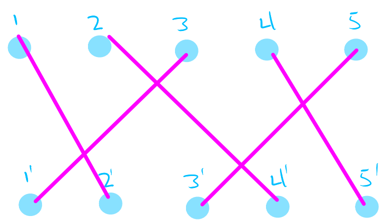


A vertex that appears in a matching is <u>saturated</u>, otherwise it is <u>unsaturated</u>.

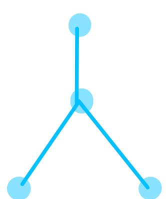
A <u>perfect matching</u> in a graph is a matching that saturates every vertex.

<u>Example</u>

In the complete bipartite graph $K_{m,n}$,



- Perfect matchings can only occur when the number of vertices is even.
- That is not a sufficient condition, as shown by the claw.



No possible perfect matching, since the center vertex is saturated by any edge.

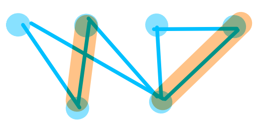
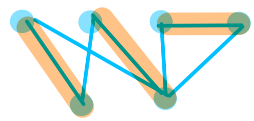
Example

Counting the perfect matchings in a complete graph.

Maximum matchings

A matching of a graph is maximal if no edge can be added. It is maximum if no other matching of this graph has more edges than it.

Example

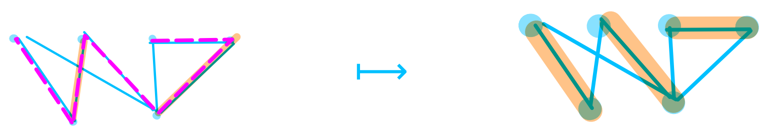
Maximal  Maximum 

Perfect Maximum Maximal

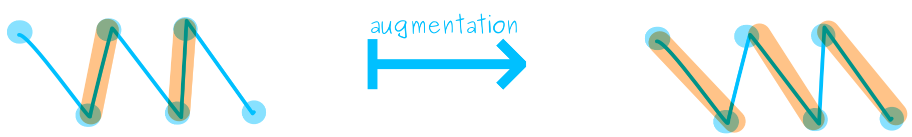
What about the converse?

Can we transform a maximal matching into a maximum matching?

M-alternating path



Let G be a graph and M be a matching of G. An M-alternating path is a path of G that alternates between edges in M and edges not in M. An M-augmenting path is an M-alternating path with both endpoints unsaturated.



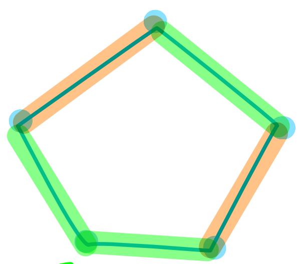
Remark: When M is maximum, there is no augmenting path.

Theorem (Berge, 1957)
A matching M in a graph is a maximum matching if and only if the graph has no M-augmenting path.

Proof
⇒ follows from the remark above
⇐ We prove the converse: if it is not maximum, it has an augmenting path. If M is not maximum, then there is a matching M' with more edges.

We consider the subgraph H with edges that appear in exactly one of M and M' (not in both).
Claim: components of H are all either even cycles or paths.
- Even cycles have as many edges from M as from M'. This is because every endpoint can have at most one edge in M and one in M'.
- Since $|M'|>|M|$, there is at least one path in H with more edges of M' than edges of M. This is a path that starts and ends with unsaturated vertices of M, so this is an M-augmenting path.

Proof of the claim: That means that every vertex of H has degree at most 2, and that cycles have even length.
The maximal degree of H is 2 by construction. At most, one vertex can have one incident edge in M and one in M'.
If a cycle has odd length, then most edges belong to the same matching, and there must be two edges belonging to the same matching and incident to the same vertex. That contradicts the construction of a matching.



problem

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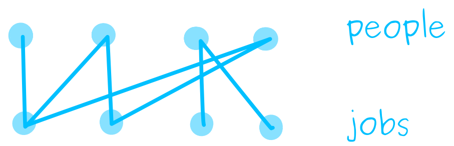
Matchings in bipartite graphs

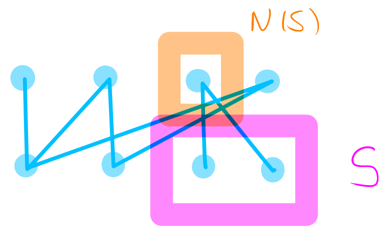
Example: Job assignments
If there are m jobs and n people, not all qualified for all the jobs, can we always fill all the jobs?



The edges are between a job and a qualified person for that job.

(The jobs cannot all be filled in this example).

Theorem (Hall's Theorem, 1935)
Let G be a bipartite graph with the independent sets X and Y forming a partition of the vertices.
G has a matching that saturates every vertex of X if and only if the neighborhood of every $S \subseteq X$ has order at least $|S|$.



Consequence: Stable marriages. Watch the video:
https://www.numberphile.com/videos/stable-marriage-problem

Reference: Douglas B. West. Introduction to graph theory, 2nd edition, 2001. Section 3.1