

How many labeled trees are there? And up to isomorphism? This is the questions we want to ask today. We will only be able to solve one of these.

Example

The number of labeled trees with $\{0,1,2,3\}$ vertices are, respectively, 1,1,1,3.

In the case of unlabeled trees, there are 1,1,1,1,2,3 trees of 0,1,2,3, 4 and 5 vertices, respectively. These are the trees up to isomorphism.

Doing the same exercise with labeled graphs, we find there are $2^{\binom{n}{2}}$ graphs. (Proof will be in homework).

So there are at most $2^{\binom{n}{2}}$ labeled trees with n vertices. But we can expect this number to be much less.

Remark: Counting trees, or graphs, up to isomorphism is incredibly difficult! Hence, we will focus on labeled trees and labeled graphs.

Theorem (Cayley's formula; proof is from Prüfer, 1918)
There are n^{n-2} labeled trees with n vertices, if $n \geq 1$.

Proof

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This proof is done using a bijection. To do so, we have to find another collection of objects indexed by the positive integers, so that there are n^{n-2} items indexed by n .

Natural choice:

Given a tree, the corresponding sequence is the Prüfer sequence or Prüfer code.

We present two algorithms: The first one takes a tree as input and transforms it into a unique n -ary sequence of length $n-2$; the second one takes an n -ary sequence of length $n-2$ and builds a unique tree with n vertices. This will prove that the number of n -ary sequences of length $n-2$ and the number of binary trees with the same vertices are the same.

Algorithm 1: Production of the sequence

Let T be a tree with vertices $\{1, 2, \dots, n\}$. If $n < 2$, we already checked that there is a single tree. So assume $n \geq 2$.

While there is more than two vertices, remove the leaf with smallest label (it always exists, by Lemma from Friday). To the sequence, append the neighbor of that leaf (it is unique, since a leaf has degree 1).

Once there are only two vertices left, stop.

Example

Algorithm 2: Production of the tree

Take a n -ary sequence s of length $n-2$, $n \geq 2$.

Draw n isolated vertices, and label them $\{1, 2, \dots, n\}$. We will add $n-1$ edges. At the beginning, no vertex is marked.

While the sequence is not empty:

Mark the smallest unmarked vertex whose label does not appear in the sequence (this always exists, since sequence only has length $n-2$).

Delete the first element of the sequence and draw an edge between this element and the vertex you just marked. This adds one edge.

When the sequence is empty, there are $n-2$ edges, one for each element of the original sequence. There are two unmarked vertices, including one isolated. Draw an edge between them, then stop.

Claim: this is a tree.

Since both algorithms are well-defined, there is a bijection between n -ary sequences of length $n-2$ and the trees with n vertices.



Note that the seemingly related problem of counting unlabeled trees is much harder... to the extent that no closed formula is known to count them. The only thing we know is an asymptotic estimate of the number of trees with n vertices when n is very big; even then, the proof is very hard and requires techniques that are far beyond the scope of this class.

Counting labeled trees with a given degree sequence

From now on, we want to count trees with n vertices, labeled $\{1, \dots, n\}$, and vertex i having degree d_i . How many such trees are there?

We use Prüfer sequences to solve this problem.

Observation: In the second algorithm, we always add an edge between the vertex we mark and the vertex that appears in the sequence (assuming we mark both vertices in the last step). So the degree of the vertex i is the number of occurrences of i in the sequence, plus 1.

Corollary

Given positive integers d_1, d_2, \dots, d_n summing to $2n-2$, there are exactly $\frac{(n-2)!}{\prod_i (d_i-1)!}$ trees with vertex set $\{1, 2, \dots, n\}$ such that vertex i has degree d_i .

Proof

