We introduce the notion of trees, a very important type of graph. Over the next week or two, we will study the properties of trees and forests.

Definition

A graph with no cycle is acyclic.
An acyclic graph is a forest; a connected forest is a tree.
A leaf is a vertex of degree 1 in a tree.

Every component here (except the one with a red X on it) is a tree, and the whole thing (without the one with the X) is a forest.

Leaves are highlighted. A star is the tree in which there is one vertex adjacent to every other.

Caveat: As graphs, trees don't need to have one specific root. We can always distinguish one root, but it is not needed. We will go back to this subject later.

Lemma

Every tree with at least two vertices has at least two leaves.
Deleting a leaf from an n-vertex tree produces a tree with $n-1$ vertices.

Proof

One consequence of that lemma is that we can build every tree with at least two vertices by "adding leaves". We will discuss that topic on Wednesday.

The following theorem gives multiple characterizations of trees:

Theorem

Let G be a graph with n vertices ($n \geq 1$). The following statements are equivalent:

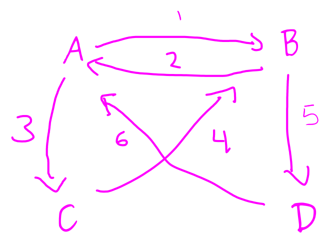
(A) G is connected and has no cycles.

(B) G is connected and has $n-1$ edges.

(C) G has no cycles and $n-1$ edges.

(D) G has no loop and has, for each pair of vertices $\{u, v\}$, exactly one uv-path.

The proof of such a statement is a closed walk that visits every vertex in the complete digraph with vertices A, B, C and D:



Proof

(1 $A \Rightarrow B$)

(2 $B \Rightarrow A$)

Until now, we proved $A \Leftrightarrow B$. They are equivalent, so we can use them together from now on.

(3 $A \Rightarrow C$)

(4 $C \Rightarrow B$)

Now, A, B and C are equivalent. That means that two characteristics among connectedness, no cycles and $n-1$ edges are sufficient to show a graph is a tree.

(5 $B \Rightarrow D$)

(6 $D \Rightarrow A$)

$\square$

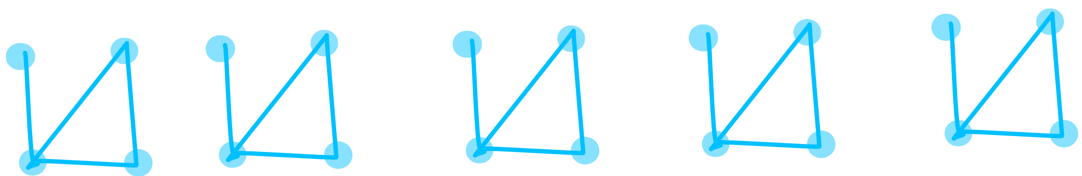
<u>Corollary</u>

a) Every edge of a tree is a cut-edge. (by A)
b) Adding one edge to a tree forms exactly one cycle (corollary of $A \Rightarrow B$).

Spanning trees

Let $G=(V,E)$ be a graph.
A graph is a <u>spanning subgraph</u> of G if it has vertex set V.
A <u>spanning tree</u> is a spanning subgraph that is a tree.

<u>Example</u>



Highlight the five spanning subgraphs of the graph in blue.

Which one are spanning trees?

Theorem

Every connected graph has a spanning tree.

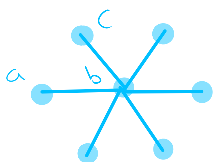
Proof

$\blacksquare$

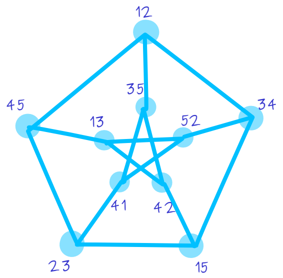
Distance in trees and graphs

If G has a uv-path, the distance between u and v, noted $d(u,v)$, is the smallest length of a uv-path. If G has no such path, $d(u,v)=\infty$.
The diameter of G, $\mathrm{diam}(G)$, is the maximum distance between two vertices.
The eccentricity of a vertex u is the distance to the furthest vertex.
The radius is the minimal eccentricity.

Example



Example

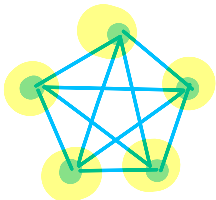


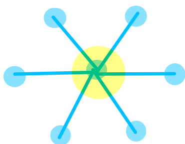
Theorem

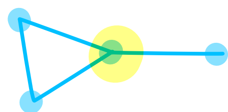
If G is a simple graph, $\mathrm{diam}(G) \geq 3 \Rightarrow \mathrm{diam}(\overline{G}) \leq 3$.

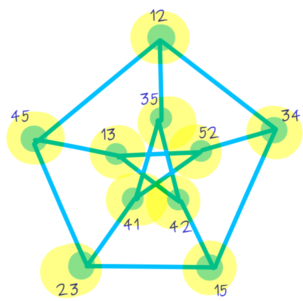
Proof: Read and understand as homework. In the book, that is Theorem 2.1.11, p.71.

The center of a graph is the induced subgraph with vertices of minimum eccentricity.





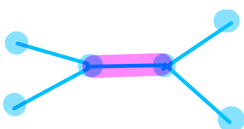




Theorem (Jordan, 1869)

The center of a tree is a vertex or an edge.

That means it cannot be a set of vertices, whenever the graph is a tree. Of course, the examples above show it is not true for graphs in general.



Reference: Douglas B. West. Introduction to graph theory, 2nd edition, 2001. Section 2.1