Math 38 — Graph Theory — Nadia Lafrenière

Extremal problems — 04/08/2022

Extremal problems consider the minimum and maximum numbers some statistics on a class of graphs can reach. We introduce some of the types of proofs useful in graph theory: Algorithmic, and by construction.

First example

In any simple graph (V,E), the maximum number of edges is

$$\binom{|V|}{2} = \frac{|V|(|V|-1)}{2}$$

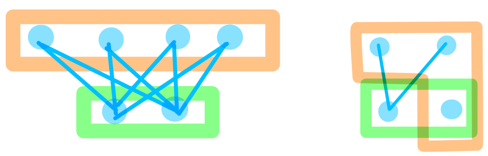
Proof

In a simple graph, there can be at most one edge per pair of distinct vertices. The maximum number of edges appear in $K_{|V|}$.

This is an extremal problem, since we are looking at the maximum number of edges. The class of graphs here is all simple graphs.

Example

In a bipartite graph with independent sets of size k and m, there can be at most km edges.



Independent sets of size 2 and 4, 8 edges at maximum. km is the number of edges of $K_{m,k}$.

Edges in connected graph

Proposition

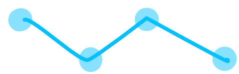
The minimum number of edges in a connected graph with n vertices is $n-1$.

Proof

We need to prove two things:

- If a graph with n vertices has fewer than $n-1$ edges, it is not connected.
- There exists a connected graph with n vertices and $n-1$ edges.

Recall from last week (Friday), that a graph with n vertices and m edges has at least $n-m$ components. Hence, if $m<n-1$, the graph has at least 2 components and is not connected.
Also, the path with n vertices has $n-1$ edges and is connected, proving that the minimum is realized.



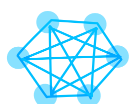
Remark (on the proof technique)
When giving the solution to an extremal problem, there are two parts to be proven:
- That the value we give is minimal (or maximal), i.e. that you cannot give a lower (respectively, higher) value.
- That this value can be realized on at least one graph of the class we consider.

Proposition
Let G be a simple graph with n vertices. If the minimum degree is $\delta(G) \geq (n-1)/2$, G is connected.

Proof
The minimum degree of the graph means that every vertex should have at least this number of neighbors, in a simple graph.
To prove that G is connected, we must show that there is a path between any pair of vertices $\{u,v\}$. We will in fact prove that there exists a path of length at most 2.
- If $\{u,v\}$ are adjacent, they are obviously in the same component.
- Otherwise, they share at least one neighbor w: There are $n-2$ other vertices, and the sum of their degree is $d(u)+d(v) \geq n-1$. Hence, $u-w-v$ is a path connecting them.

A bound is said to be sharp if improving it (reducing a lower bound or increasing an upper bound) would make the statement wrong.
The bound in the last problem is sharp. To prove it, we give an example of a graph with n vertices and minimum degree $\lfloor \frac{n}{2} \rfloor - 1$ that is not connected: This graph is the disjoint union of $K_{\lfloor \frac{n}{2} \rfloor}$ and $K_{\lceil \frac{n}{2} \rceil}$.

K_5, degree 4



K_6, degree 5

11 vertices
Minimum degree is 4, just under $5 = (11-1)/2$.
Graph is disconnected.

Bipartite subgraph

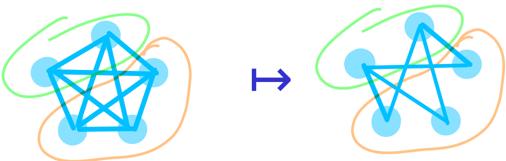
Here we prove that, given a graph G, we can always find a bipartite subgraph with at least a fixed number of edges. We give an algorithmic proof to construct the graph, but a proof can also be done by induction.

Theorem

Every loopless graph $G=(V,E)$ has a bipartite subgraph with at least $|E|/2$ edges.

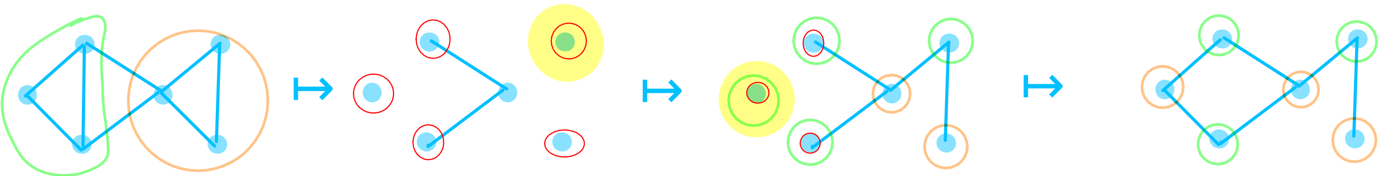
Proof (algorithmic)

We start with any partition of the vertices into two sets X and Y. Let H be the subgraph containing all the vertices, but only the edges with one endpoint in X and one in Y.



6 edges, instead of 10

Let v be a vertex in X. If H has fewer than half the edges incident to v, then it means that v has (in G) more neighbors in X than in Y. To increase the number of edges in H, switch v to Y. The number of edges just increased.

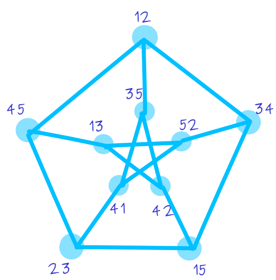


○ = less than half the edges

As long as H does not have at least half the edges of G at every vertex, there are vertices that can be swapped from X to Y or Y to X; repeat this process. When it terminates, the number of edges in H is always at least half the number of edges of G. ∎

Triangle-free graphs

A graph is said to be <u>triangle-free</u> if it has no three vertices that are all adjacent. In general, a graph G is H-free if it does not contain H as a subgraph.



The Petersen graph is triangle-free (but not bipartite).

Theorem (Mantel, 1907)

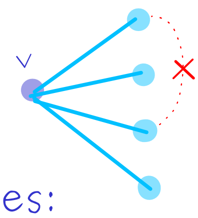
The maximum number of edges in a simple triangle-free graph with n vertices is $\lfloor \frac{n^2}{4} \rfloor$.

Proof

For the proof, we again need to prove two things:

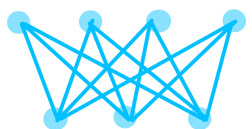
- that a triangle-free graph with n vertices cannot have more than $\lfloor \frac{n^2}{4} \rfloor$ edges.
- that there exists, for any n, a graph with n vertices and $\lfloor \frac{n^2}{4} \rfloor$ edges that has no triangle.

For the first part, assume the graph is triangle-free. Take a vertex v of maximal degree Δ. Its Δ neighbors cannot have edges among them. So every edge of G must have at least one endpoint in a non-neighbor of v, or in v itself. There are $n-\Delta$ such vertices. Each such vertex has degree as most Δ.



Therefore, we give an upper bound on the number of edges: the number of edges is at most $\Delta(n-\Delta)$ (because $n-\Delta$ is the number of vertices not adjacent to v). Maximizing $\Delta(n-\Delta)$ gives $\Delta = n/2$. Hence, the number of edges is at most $\lfloor \frac{n^2}{4} \rfloor$.

For the second part, we must prove that a triangle-free graph has $\lfloor \frac{n^2}{4} \rfloor$ edges. This is the case of $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$.



We can split 7 vertices into two sets of 3 and 4 vertices, which leads to 12 edges:, which is the smallest integer below 49/4.

Reference: Douglas B. West. Introduction to graph theory, 2nd edition, 2001. Section 1.3