Math 38 — Graph Theory
Bipartite and Eulerian Graphs

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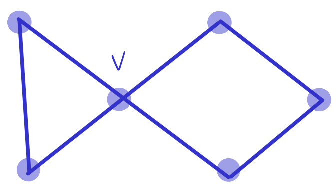
Today's lecture aims to give the important properties of bipartite graphs. We will also define Eulerian circuits and Eulerian graphs: this will be a generalization of the Königsberg bridges problem.

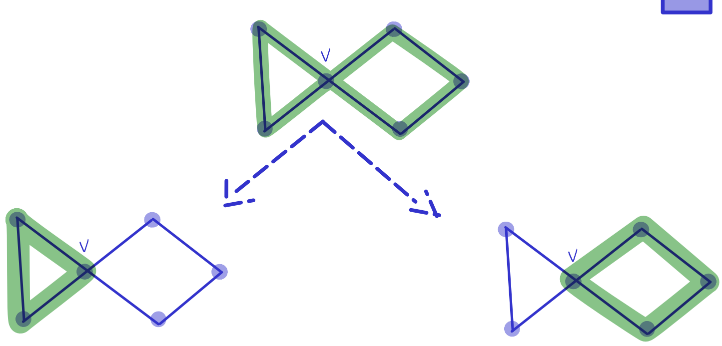
Characterization of bipartite graphs

The goal of this part is to give an easy test to determine if a graph is bipartite using the notion of cycles: König theorem says that a graph is bipartite if and only if it has no odd cycle.

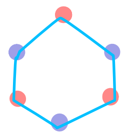
Lemma
Every closed walk of odd length contains an odd cycle. This is called an odd closed walk.

Proof
We prove it using strong induction on the length of the walk (i.e. the number of edges).
Base case: length 1. The walk is a loop, which is an odd cycle.
Induction hypothesis: If a walk has odd length at most n, then it contains an odd cycle.
Induction step: Consider a closed walk of odd length n+1. If it has no repeated vertex (except the first and last one), this is a cycle of odd length. Otherwise, assume vertex v is repeated. We can split the walk into two closed walks starting and ending at v, one of even length, and one of odd length smaller than n. By induction hypothesis, the latter contains an odd cycle. $\blacksquare$





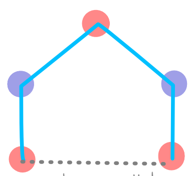
That lemma will be helpful for characterizing bipartite graphs. Of course, bipartite graphs can have even cycles, which starts in one independent set and ends there.



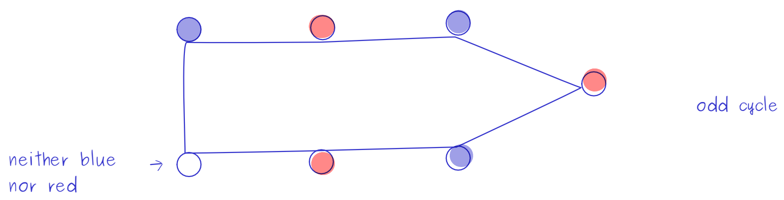
We can represent the independent sets using colors.

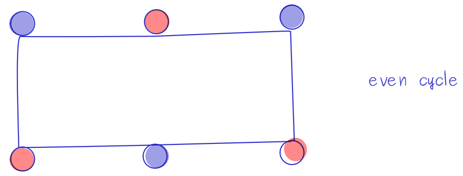
<u>Theorem</u> (König, 1936)
A graph is bipartite if and only if it has no odd cycle.

<u>Proof</u>
Notice that a graph is bipartite if and only if all its components are bipartite. So we do the proof on the components.
⇒ We prove the contrapositive: it is has an odd cycle, it is not bipartite.
Since every cycle must end at the vertex where it starts, it starts and ends in the same independent set. Since every edge is going from one set to the other, we alternate between the two sets. At the end of the cycle, we cannot close it, since we would need to change the set of the first vertex. Hence, if a connected graph is bipartite, it has no odd cycle.



⇐ We still need to prove that a connected graph without odd cycle is bipartite. If the graph has only one vertex, it is bipartite.
Otherwise, start at vertex u, and color its neighbors with color blue. Then, color the neighbors of the blue vertices in red, and repeat this process by coloring the neighbors of the red vertices in blue, until all vertices have been colored. I claim that no vertex will change color in that process; assume otherwise, that v is changing color. That would mean that there exists a path of odd length from u to v (the one that colors v in blue), and a path of even length doing it (the one that colors v in red). The combination of these two paths is an odd walk, and contains an odd cycle, which is prohibited by the hypothesis. Hence, the coloring is well defined, and the two colors represent independent sets. The graph is bipartite. ∎





Technique for checking whenever a graph is bipartite:
- If it is bipartite, prove it by finding two independent sets.
- If it is not bipartite, find an odd cycle.

Eulerian circuits

A graph is Eulerian if it has a closed trail containing all the edges.

The graph in the Königsberg bridges problem is not Eulerian. We saw that the fact that some vertices had odd degree was a problem, since we could never return to that vertex after leaving it for the last time.

Theorem

A graph is Eulerian if and only if it has at most one nontrivial component (i.e. component with edges), and if every vertex has even degree.

Proof

We first prove ⇒ by proving the contrapositive: if a graph has more than one non-trivial component, or it there is a vertex of odd degree, then the graph is not Eulerian.

If a graph has at least two non-trivial components, there can't be a walk going through all the edges, since they are in separate components.

If a graph has a vertex of odd degree, we are in the case of the Königsberg bridges: we can leave the vertex more often than we can come back (or vice-versa), and thus our trail cannot be closed.

⇐ We need to prove that a connected graph with only vertices of even degrees is Eulerian. We can ignore the isolated vertices for this since we are focusing on the edges. The following lemma is useful:

Lemma

If every vertex of a graph has degree at least 2, then it contains a cycle.

Proof

Let P be a maximal path in that graph. If it is a cycle, we are done. Otherwise, let u be an endpoint of P. Since it has degree at least 2, u has a neighbor v not in P. But since P is maximal, that means that v is already in P, and the edge uv completes the cycle.

Proof of the theorem (continued)

We proceed by induction on the number of edges.

Base case: 0 edge, the graph is Eulerian.

Induction hypothesis: A graph with at most n edges is Eulerian.

Induction step: If all vertices have degree 2, the graph is a cycle (by definition) and it is Eulerian. Otherwise, let G' be the graph obtained by deleting a cycle. The lemma we just proved shows it is always possible to delete a cycle. By induction hypothesis, G' is Eulerian. To build an Eulerian circuit in G, start by the cycle we just deleted, and append the Eulerian circuit of G'. ∎

Proposition

Every graph with only vertices of even degree decomposes into cycles.

Eulerian circuits are closed trails that pass through all edges. A similar property is being Hamiltonian: a Hamiltonian circuit is a circuit that passes though all vertices exactly once. A Hamiltonian graph is a graph with a Hamiltonian circuit.

Reference: Douglas B. West. Introduction to graph theory, 2nd edition, 2001. Section 1.2