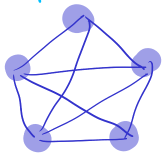
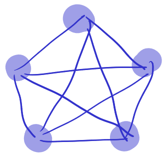
Math 38 — Graph Theory Nadia Lafrenière

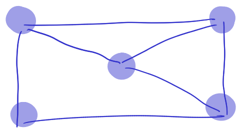
Basic definitions and some problems 03/28/2022

Can you draw these pictures, without ever crossing your path?



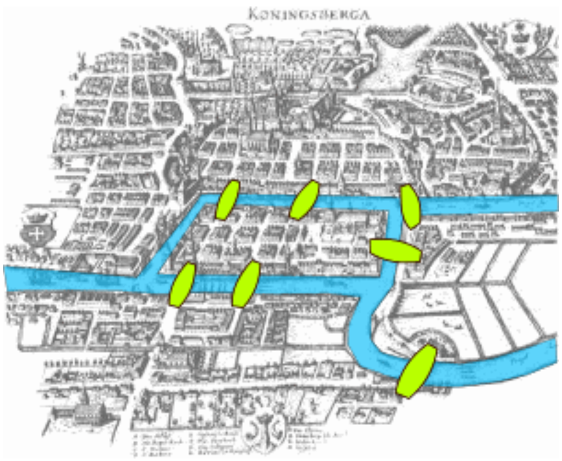


Can you draw this picture without ever lifting your pencil?



These are children problems, but also real-life problems in graph theory, namely to know whether a graph is planar, or similar to know if a graph is Eulerian.

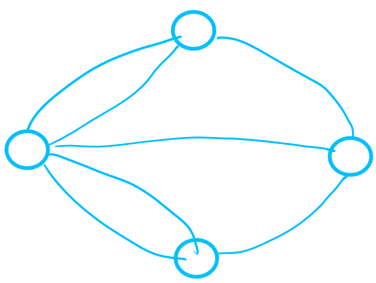
The first problem: Seven bridges of Königsberg (Euler, 1736)



Euler was wondering if one can go from one place in the Königsberg area, and back to that original place, by taking every bridge exactly once.

(This is considered to be the first solved problem in graph theory).

A modelisation of the problem:



This graph model the areas of the city. There is no need to know the exact location of each bridge.

Remarks:

- Since we have to go back where we started, we do not care where we start.
- Everytime we go from a location to another and back, we cross 2 bridges adjacent to that location.

Since every island has an odd number of bridges, it is not possible to visit all the islands by taking every bridge exactly once.

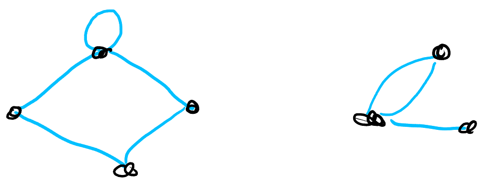
Some definitions

A graph G is made of a set of vertices (modeling some objects), and a set of relations between two vertices, called the edges. We denote $G = (V,E)$ for the graph with vertices V and edges E. Any edge is a pair of two vertices called the endpoints.

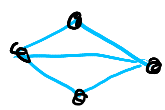
We draw a graph (on paper or on the computer) by representing the vertices as points, and we draw a curve between two vertices if they are endpoints of the same edge. We can draw differently the same graph.

Example

A loop is an edge whose endpoints are the same vertex. 
Multiple edges are edges having the same pair of endpoints.
A simple graph is a graph having no loop nor multiple edges.



Not simple graphs



Simple graph

When uv (or equivalently) vu is an edge, we say the vertices u and v are adjacent, or that they are neighbors.

Subgraphs and containment

A graph $G'=(V',E')$ is a subgraph of $G=(V,E)$ if $V' \subseteq V$ and $E' \subseteq E$.
We then say that G' is contained in G, denoted $G' \subseteq G$.

Example
Every graph with n vertices is a subgraph of the complete graph with $m \geq n$ vertices.

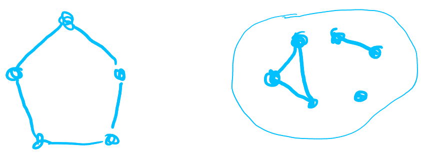
A graph is connected if, for every pair of vertices, there is a path (i.e. a sequence of edges) between them that belongs to the graph. It is otherwise disconnected.

Some important problems in graph theory

1. Acquaintances

Do every set of six people contain at least three mutual acquaintances or three mutual strangers?

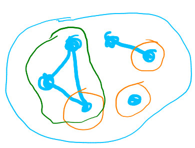
That question can be represented using a graph. Every person is a vertex, and there is an edge between two persons if they know each other. Here, we assume knowing each other is a mutual relation, i.e. knowing a celebrity usually does not count.



Two graphs. The first one is a 5-vertex graph with no three mutual strangers, nor three acquaintances. The second one has six vertices, and contain both three mutual strangers and three acquaintances (a clique).

As a homework, you will have to prove your solution to this statement.

Some useful vocabulary:
A clique in a graph is a set of pairwise adjacent vertices, i.e. a complete subgraph.
An independent set is a subset of vertices with no adjacent pairs.

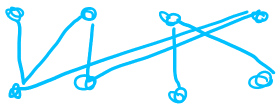


- A clique
- An independent set

2. Job assignments

If there are m jobs and n people, not all qualified for all the jobs, is there a way we can fill all the jobs?

Definition
A bipartite graph is the disjoint union of two independent sets.



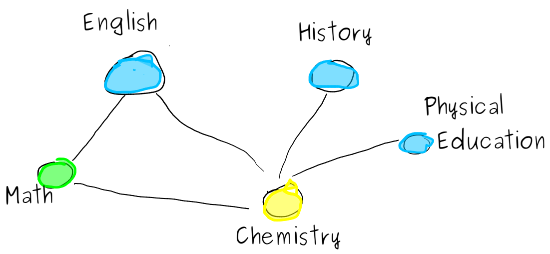
people
jobs

The edges are between a job and a qualified person for that job.

(The jobs cannot all be filled in this example).

3. Scheduling and avoiding conflicts

My high school used to have a very long exam sessions at the end of the year, and there were still some conflicts. I wish the administrators knew graph theory...



Vertices: Subjects

Edges: If someone takes both subjects, i.e. eventual scheduling conflicts.

A coloring of a graph is a partition of a set into independent sets. Scheduling with no conflicts is equivalent to coloring. If we want to use the minimum time, we should use as few colors as possible.

Schedule:

1. History—English—PE
2. Chemistry
3. Math

Reference: Douglas B. West. Introduction to graph theory, 2nd edition, 2001. Sections 1.1.1 and 1.1.2.

Math 38 — Graph Theory

Nadia Lafrenière

Matrices for graph and Isomorphisms

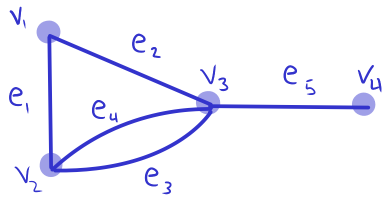
03/30/2022

We saw last class that two graphs are the same if they are differently, as long as we are simply "moving the vertices". The goal of today's lecture is to make this statement more formal. One tool we will use is adjacency and incidence matrices. We will as well start classifying the graphs.

Matrices: adjacency matrix and incidence matrix

Let $G=(V, E)$ be a graph without any loop (it does not have to be a simple graph). We number the vertices from 1 to n and the edges from 1 to m.

The adjacency matrix of G, written $A(G)$, is the matrix whose (i,j)-entry is the number of edges with endpoints the vertices i and j.
The incidence matrix of G, written $M(G)$, is the n-by-m matrix whose (i,j)-entry is 1 if vertex i is an endpoint of edge j, and otherwise 0.
The adjacency matrix is always a symmetric matrix.
The graph on the left has the following adjacency and incidence matrices:



$$A(G) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad M(G) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The degree of a vertex (in a loopless graph) is the number of edges incident to that vertex.

Isomorphisms

So when are two graphs the same? We will answer this question using the notion of a bijection. As a reminder, this an injective and surjective function, or a one-to-one correspondence.

An isomorphism from a simple graph G to a simple graph H is a bijection $f: V(G) \to V(H)$ such that every edge uv of G is mapped to the edge $f(u)f(v)$ of H. We then say G and H are isomorphic, denoted $G \cong H$.

This is equivalent to asking that there exists a simultaneous permutation of the rows and columns of the adjacency matrix of G that would yield the adjacency matrix of H.

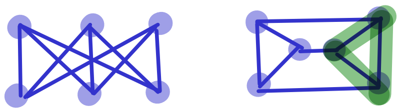
Example
The following graphs are isomorphic:

This is easily seen with the bijection that exchanges 1 and 3.

Remarks:

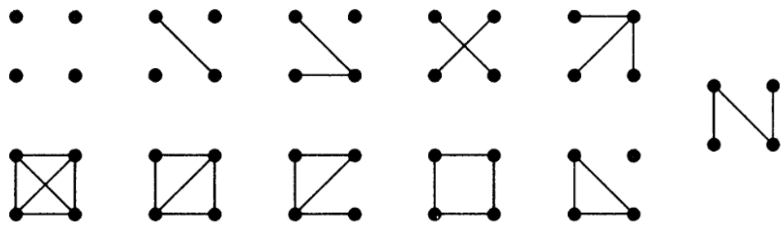
- Finding a bijection of the labels is the way to prove two graphs are isomorphic. However, to prove they are not isomorphic, there are many ways. For example, if the list of degrees is not the same, you will never be able to find an isomorphism. Or if the number of edges (or edges) do not correspond. Among others.
- The isomorphism relation is an equivalence relation, i.e. this is a symmetric relation ($G \cong H$ iff $H \cong G$), a transitive relation ($G \cong H$ and $H \cong J$ imply $G \cong J$) and a reflexive one ($G \cong G$). That means that we can split the graph into equivalence classes.

Example
The following graphs are not isomorphic. They both have six vertices, all of degree 3, and nine edges, and they are both connected, but one is bipartite and the other is not. Since they don't have the same properties, they are not isomorphic.



No triangle appear in the first graph.

Example
All the isomorphism classes for graphs with 4 vertices are



Special graphs

There are some graphs that have special names, and that turns out to be handy for whenever we want to use them or to classify them.

Complete graphs: Graphs with n vertices and $\binom{n}{2}$ edges.

Example: K_5 

K_n

Complete bipartite graphs: Bipartite graphs with independent sets of size s and r, with sr edges.

$K_{s,r}$

Example: $K_{4,2}$ 

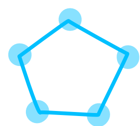
Paths: Connected graphs, with all the vertices of degree 2, except at most two who have degree 1.

P_n

Example: P_4 

Cycles: Paths with as many edges as vertices.

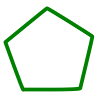
C_n

Example: C_5 

The complement of the graph G is the graph that has the same vertices and whose edges are all the edges that do not belong to G: $K_{|V|} - E(G) = \overline{G}$.

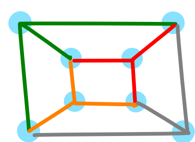
A graph G is self-complementary if its complement $\overline{G}$ is isomorphic to G.

Example: C_5 is self-complementary.

$K_5 =$   $\cong$ 

A decomposition of a graph is a list of subgraphs in which every edge appears exactly once.

Example: The cube decomposed into copies of $K_{1,3}$



Note: $K_{1,3}$ is often called the claw.

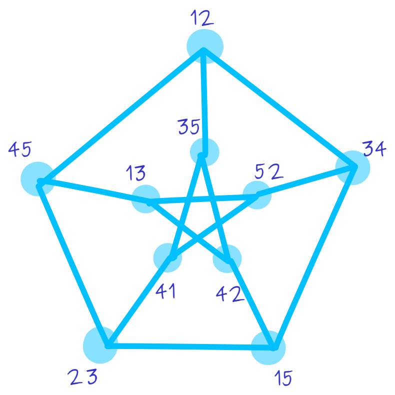
Proposition

A graph G is self-complementary if and only if the complete graph is a decomposition into two copies of G.

The Petersen graph

The Petersen graph is a 10-vertices graph with 15 edges that is very famous, as it is an example or a counter-example to many phenomena.

The Petersen graph is the graph of 2-element subsets of $\{1,2,3,4,5\}$, and there is an edge between 2 subsets if their intersection is empty.



Some properties of the Petersen graph:

- Two non-adjacent vertices share exactly one neighbor.
- The graph has no triangle, but is not bipartite.
- The shortest cycle in the Petersen graph has length 5.

(The length of the shortest cycle in a graph is called the girth of the graph.)

Reference: Douglas B. West. Introduction to graph theory, 2nd edition, 2001. Section 1.1

Math 38 — Graph Theory Nadia Lafrenière

Connection in graphs 04/01/2022

Today's lecture aims to define the proper vocabulary to talk about trajectories and connectedness in graphs.

Definitions

Recall that a path is a graph whose vertices can be ordered without repetition (except maybe for the endpoints) in a sequence such that two consecutive vertices are adjacent. A path is a u,v-path if it starts at vertex u and ends at vertex v.

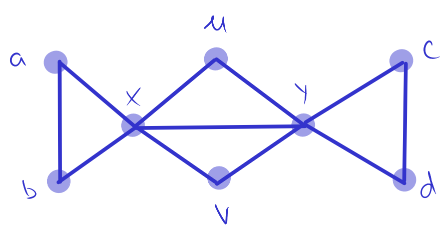
A walk is a list $(v_0, e_1, v_1, \dots, e_k, v_k)$ of vertices and edges such that the edge e_i has endpoints v_{i-1} and v_i. A walk is a u,v-walk if its endpoints (the first and last vertices of the walk) are u and v. If there is no multiple edges, we can write the walk as $(v_0, v_1, \dots, v_k)$.

A trail is a walk with no repeated edge. Similarly, a u,v-trail has endpoints u and v.

The points that are not endpoints are internal vertices.

The length of a walk, trail, path or cycle is its number of edges.
A walk or a trail is closed if its endpoints are the same.

Example



(a,x,a,b,x,u,y,x,a) specifies a closed walk, but not a trail (ax is used more than once).

(a,b,x,u,y,x,a) specifies a closed trail.

The graph contains the five cycles (a,b,x,a), (u,y,x,u), (v,y,x,v), (x,u,y,v,x) and (y,c,d,y).

The trail (x,u,y,c,d,y,v,x) is not an example of a cycle, since vertex y is repeated (so it is not a path).

Lemma

Every u,v-walk contains a u,v-path.

Proof

The proof can be done using the principle of strong induction, and we induce on the number of edges.

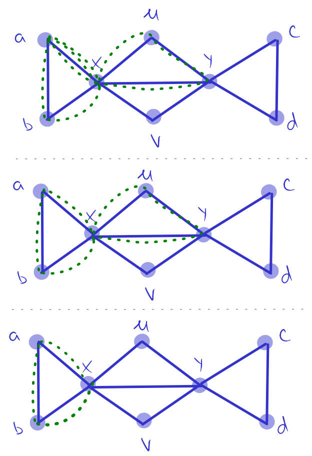
Base case: No edge, $u=v$ is the only vertex in the graph. Only walk has length 0, and is therefore a path.

Induction hypothesis: Assume that, for a walk with $k<n$ edges, there is always a path with the same endpoints.

Induction step: The walk has n edges. There are two cases: either there is no repeated vertex or only the endpoint is repeated, and then the walk is already a path, or there is a repeated vertex x. In the latter case, we delete the edges between the first and last occurrences of x, which leaves us with only one copy of x, and a u,v-walk with fewer than n edges. We can thus use the induction hypothesis to conclude that there exists a u,v-path in the u,v-walk.

■

Example: The u,v-walk from previous page.



In the walk (a,x,a,b,x,u,y,x,a), we delete what happens between the first two occurrences of a, and get the closed walk (a,b,x,u,y,x,a). Then we delete what happens between the two occurrences of x, and get the cycle (a,b,x,a), which is a path.

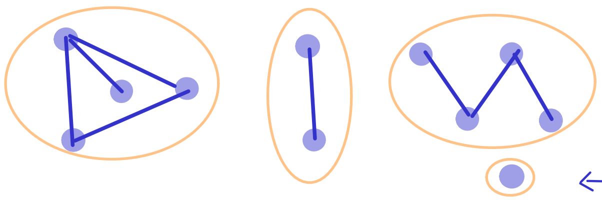
Connectedness, components and cuts

Recall that a graph is connected if and only if there exists a path between u and v for every pair of vertices $\{u,v\}$.

A <u>component</u> of a graph G is a maximal connected subgraph.
A component is <u>trivial</u> if it has no edges; in this case, the unique vertex is said to be an <u>isolated vertex</u>.

Example

The following graph has 4 components, each of which are circled in orange.



Proposition

Every graph with n vertices and k edges has at least $n-k$ components.

Proof

The proof can be done by induction on k. The case of $k>n$ is obvious, since the number of components is always nonnegative.

Base case: If $k=0$, then each of the n vertices are isolated, and there are n components.

Induction hypothesis: Assume that a graph with $k-1$ edges and n vertices has at least $n-k+1$ components.

Induction step: Let $G=(V,E)$ with $|V|=n$ and $|E|=k$. Remove the edge e to get $G-e$. The component of G containing e can either be split into two components by removing e, or stay a component. So G has either the same number of components as $G-e$, or one fewer. By induction hypothesis, $G-e$ has at least $n-k+1$ components, so G has at least $n-k$.

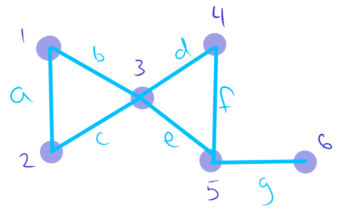
∎

In the last proof, we had to distinguish the cases where removing the edge was creating a new component or not. An edge whose deletion creates new component has a special name:

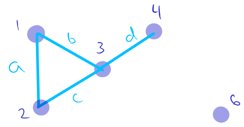
A cut-edge or cut-vertex of a graph is an edge or vertex whose deletion increases the number of components. We write $G-e$ or $G-M$ for the subgraph of G obtained by deleting an edge e or a set of edges M; we write $G-v$ and $G-S$ for the graph obtained by deleting a vertex v or a set of vertices S along with their incident edges.

A subgraph obtained by deleting a subset of vertices and their incident edges is an induced subgraph: we denote it $G[T]$ if $T=V\setminus S$ and we deleted the vertices in S.

Example



Vertices 3 and 5 are cut-vertices, and the edge g is the only cut-edge.
The induced subgraph for the vertices 1, 2, 3, 4 and 6:



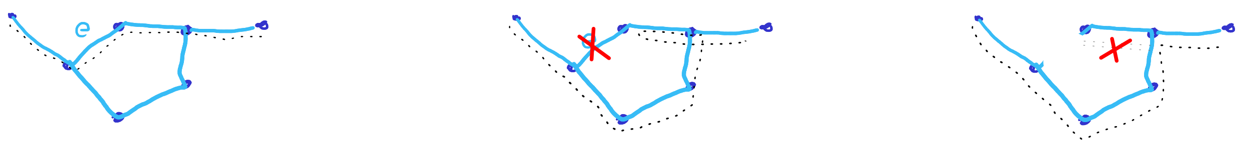
Theorem

An edge is a cut-edge if and only it if belongs to no cycle.

Proof

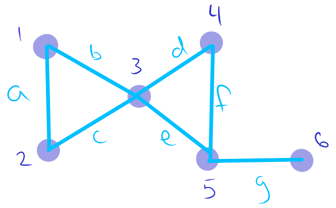
Let $e=uv$ be an edge in the graph G, and let H be the component containing e. We can restrict the proof to H, since deleting e does not influence the other components. We want to prove that $H-e$ is connected if and only if e is in a cycle in H.

If e is in a cycle c, $c-e$ is a path P between v and u avoiding the edge e. To show that $H-e$ is still connected, we need to show that, for every pair of vertices $\{x,y\}$, there is a path between x and y. Since H is connected, there exists in H such a path. If that path does not contain e, it is still in $H-e$. Otherwise, replace e by P, and remove an edge from that path everytime it appears twice consecutively.

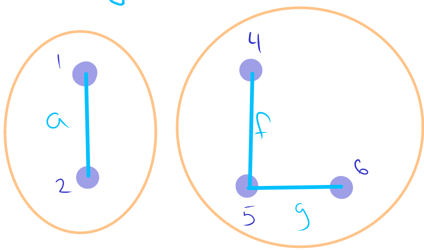


If $H-e$ is connected, then there exists in it a path P between u and v. Hence, adding edge $e=uv$ creates the cycle $P+e$. ∎

The last theorem allows us to characterize cut-edges. Would such a theorem be possible for cut-vertices? The following example proves that asking for it to be outside a cycle is not a requirement for a cut-vertex, since vertex 3 is a cut-vertex, and belongs to two cycles:



Removing vertex 3:



Two connected components

Reference: Douglas B. West. Introduction to graph theory, 2nd edition, 2001. Section 1.2