Math 38 — Graph Theory

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Connection in graphs

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Today's lecture aims to define the proper vocabulary to talk about trajectories and connectedness in graphs.

Definitions

Recall that a path is a graph whose vertices can be ordered without repetition (except maybe for the endpoints) in a sequence such that two consecutive vertices are adjacent. A path is a u,v–path if it starts at vertex u and ends at vertex v.

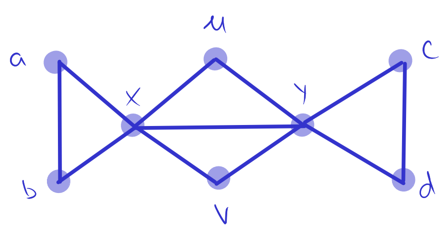
A walk is a list $(v_0, e_1, v_1, \ldots, e_k, v_k)$ of vertices and edges such that the edge e_i has endpoints v_{i-1} and v_i. A walk is a u,v–walk if its endpoints (the first and last vertices of the walk) are u and v. If there is no multiple edges, we can write the walk as $(v_0, v_1, \ldots, v_k)$.

A trail is a walk with no repeated edge. Similarly, a u,v–trail has endpoints u and v.

The points that are not endpoints are internal vertices.

The length of a walk, trail, path or cycle is its number of edges.
A walk or a trail is closed if its endpoints are the same.

Example



(a,x,a,b,x,u,y,x,a) specifies a closed walk, but not a trail (ax is used more than once).

(a,b,x,u,y,x,a) specifies a closed trail.

The graph contains the five cycles (a,b,x,a), (u,y,x,u), (v,y,x,v), (x,u,y,v,x) and (y,c,d,y).

The trail (x,u,y,c,d,y,v,x) is not an example of a cycle, since vertex y is repeated (so it is not a path).

Lemma

Every u,v-walk contains a u,v-path.

Proof

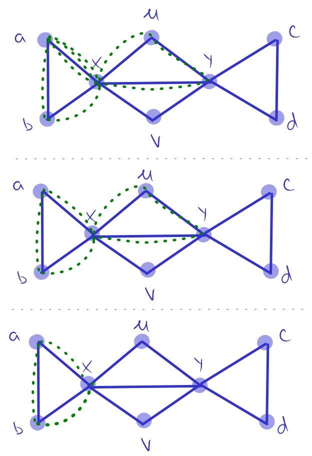
The proof can be done using the principle of strong induction, and we induce on the number of edges.

Base case: No edge, $u=v$ is the only vertex in the graph. Only walk has length 0, and is therefore a path.

Induction hypothesis: Assume that, for a walk with $k<n$ edges, there is always a path with the same endpoints.

Induction step: The walk has n edges. There are two cases: either there is no repeated vertex or only the endpoint is repeated, and then the walk is already a path, or there is a repeated vertex x. In the latter case, we delete the edges between the first and last occurrences of x, which leaves us with only one copy of x, and a u,v-walk with fewer than n edges. We can thus use the induction hypothesis to conclude that there exists a u,v-path in the u,v-walk.

Example: The u,v-walk from previous page.



In the walk (a,x,a,b,x,u,y,x,a), we delete what happens between the first two occurrences of a, and get the closed walk (a,b,x,u,y,x,a). Then we delete what happens between the two occurrences of x, and get the cycle (a,b,x,a), which is a path.

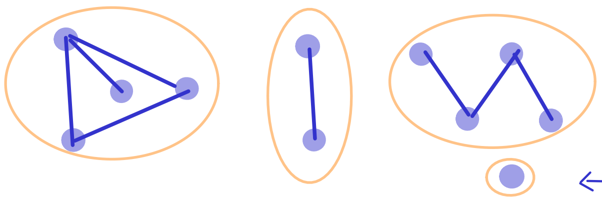
Connectedness, components and cuts

Recall that a graph is connected if and only if there exists a path between u and v for every pair of vertices $\{u,v\}$.

A component of a graph G is a maximal connected subgraph.
A component is trivial if it has no edges; in this case, the unique vertex is said to be an isolated vertex.

Example

The following graph has 4 components, each of which are circled in orange.



Proposition

Every graph with n vertices and k edges has at least $n-k$ components.

Proof

The proof can be done by induction on k. The case of $k>n$ is obvious, since the number of components is always nonnegative.

Base case: If $k=0$, then each of the n vertices are isolated, and there are n components.

Induction hypothesis: Assume that a graph with $k-1$ edges and n vertices has at least $n-k+1$ components.

Induction step: Let $G=(V,E)$ with $|V|=n$ and $|E|=k$. Remove the edge e to get $G-e$. The component of G containing e can either be split into two components by removing e, or stay a component. So G has either the same number of components as $G-e$, or one fewer. By induction hypothesis, $G-e$ has at least $n-k+1$ components, so G has at least $n-k$.

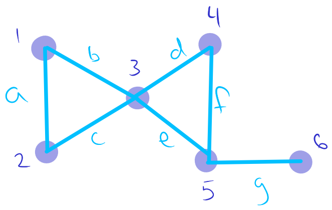
$\blacksquare$

In the last proof, we had to distinguish the cases where removing the edge was creating a new component or not. An edge whose deletion creates new component has a special name:

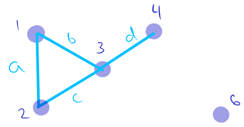
A <u>cut-edge</u> or <u>cut-vertex</u> of a graph is an edge or vertex whose deletion increases the number of components. We write $G-e$ or $G-M$ for the subgraph of G obtained by deleting an edge e or a set of edges M; we write $G-v$ and $G-S$ for the graph obtained by deleting a vertex v or a set of vertices S along with their incident edges.

A subgraph obtained by deleting a subset of vertices and their incident edges is an induced subgraph: we denote it G[T] if T=V\S and we deleted the vertices in S.

Example



Vertices 3 and 5 are cut-vertices, and the edge g is the only cut-edge.
The induced subgraph for the vertices 1, 2, 3, 4 and 6:



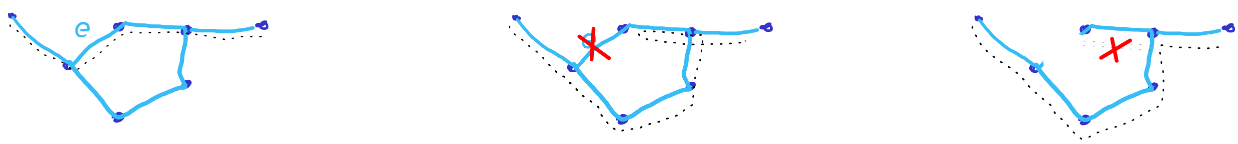
Theorem

An edge is a cut-edge if and only it if belongs to no cycle.

Proof

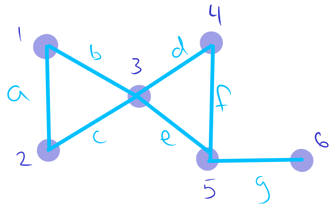
Let e=uv be an edge in the graph G, and let H be the component containing e. We can restrict the proof to H, since deleting e does not influence the other components. We want to prove that H−e is connected if and only if e is in a cycle in H.

If e is in a cycle c, c−e is a path P between v and u avoiding the edge e. To show that H−e is still connected, we need to show that, for every pair of vertices {x,y}, there is a path between x and y. Since H is connected, there exists in H such a path. If that path does not contain e, it is still in H−e. Otherwise, replace e by P, and remove an edge from that path everytime it appears twice consecutively.

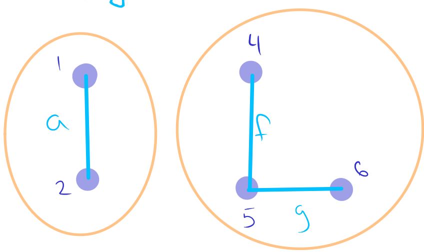


If H−e is connected, then there exists in it a path P between u and v. Hence, adding edge e=uv creates the cycle P+e. ∎

The last theorem allows us to characterize cut-edges. Would such a theorem be possible for cut-vertices? The following example proves that asking for it to be outside a cycle is not a requirement for a cut-vertex, since vertex 3 is a cut-vertex, and belongs to two cycles:



Removing vertex 3:



Two connected components

Reference: Douglas B. West. Introduction to graph theory, 2nd edition, 2001. Section 1.2