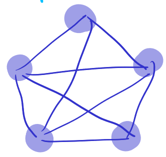
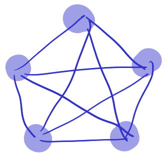
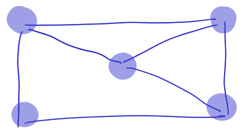
Math 38 — Graph Theory Nadia Lafrenière

Basic definitions and some problems 03/28/2022

Can you draw these pictures, without ever crossing your path?

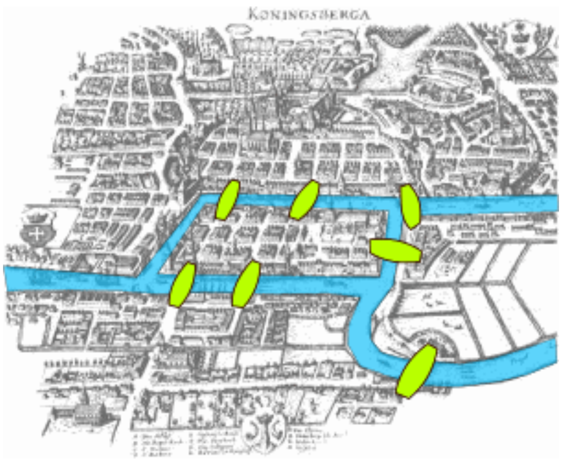
 

Can you draw this picture without ever lifting your pencil?



These are children problems, but also real-life problems in graph theory, namely to know whether a graph is planar, or similar to know if a graph is Eulerian.

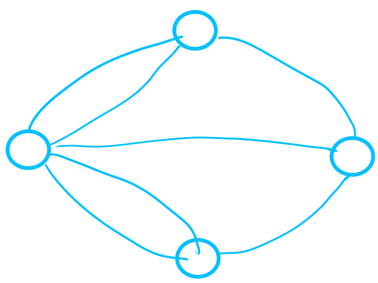
The first problem: Seven bridges of Königsberg (Euler, 1736)



Euler was wondering if one can go from one place in the Königsberg area, and back to that original place, by taking every bridge exactly once.

(This is considered to be the first solved problem in graph theory).

A modelisation of the problem:



This graph model the areas of the city. There is no need to know the exact location of each bridge.

Remarks:

- Since we have to go back where we started, we do not care where we start.
- Everytime we go from a location to another and back, we cross 2 bridges adjacent to that location.

Since every island has an odd number of bridges, it is not possible to visit all the islands by taking every bridge exactly once.

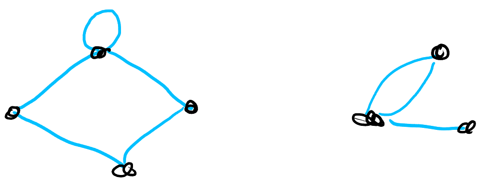
Some definitions

A graph G is made of a set of vertices (modeling some objects), and a set of relations between two vertices, called the edges. We denote $G = (V,E)$ for the graph with vertices V and edges E. Any edge is a pair of two vertices called the endpoints.

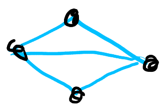
We draw a graph (on paper or on the computer) by representing the vertices as points, and we draw a curve between two vertices if they are endpoints of the same edge. We can draw differently the same graph.

Example

A loop is an edge whose endpoints are the same vertex. 
Multiple edges are edges having the same pair of endpoints.
A simple graph is a graph having no loop nor multiple edges.



Not simple graphs



Simple graph

When uv (or equivalently) vu is an edge, we say the vertices u and v are adjacent, or that they are neighbors.

Subgraphs and containment

A graph $G'=(V',E')$ is a subgraph of $G=(V,E)$ if $V' \subseteq V$ and $E' \subseteq E$.
We then say that G' is contained in G, denoted $G' \subseteq G$.

Example

Every graph with n vertices is a subgraph of the complete graph with $m \geq n$ vertices.

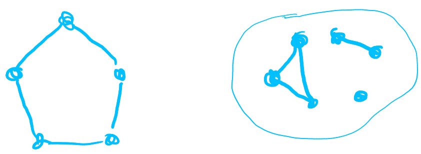
A graph is connected if, for every pair of vertices, there is a path (i.e. a sequence of edges) between them that belongs to the graph. It is otherwise disconnected.

Some important problems in graph theory

1. Acquaintances

Do every set of six people contain at least three mutual acquaintances or three mutual strangers?

That question can be represented using a graph. Every person is a vertex, and there is an edge between two persons if they know each other. Here, we assume knowing each other is a mutual relation, i.e. knowing a celebrity usually does not count.



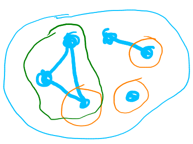
Two graphs. The first one is a 5-vertex graph with no three mutual strangers, nor three acquaintances. The second one has six vertices, and contain both three mutual strangers and three acquaintances (a clique).

As a homework, you will have to prove your solution to this statement.

Some useful vocabulary:

A clique in a graph is a set of pairwise adjacent vertices, i.e. a complete subgraph.

An independent set is a subset of vertices with no adjacent pairs.



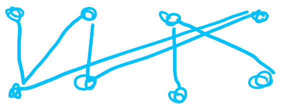
- A clique
- An independent set

2. Job assignments

If there are m jobs and n people, not all qualified for all the jobs, is there a way we can fill all the jobs?

Definition

A bipartite graph is the disjoint union of two independent sets.

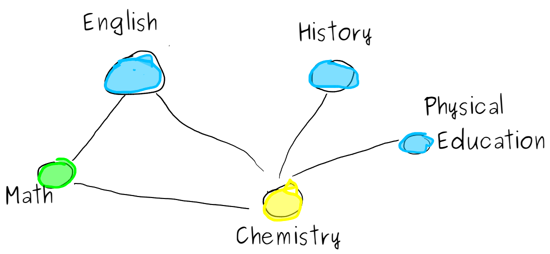


The edges are between a job and a qualified person for that job.

(The jobs cannot all be filled in this example).

3. Scheduling and avoiding conflicts

My high school used to have a very long exam sessions at the end of the year, and there were still some conflicts. I wish the administrators knew graph theory...



Vertices: Subjects

Edges: If someone takes both subjects, i.e. eventual scheduling conflicts.

A coloring of a graph is a partition of a set into independent sets. Scheduling with no conflicts is equivalent to coloring. If we want to use the minimum time, we should use as few colors as possible.

Schedule:

1. History–English–PE
2. Chemistry
3. Math

Reference: Douglas B. West. Introduction to graph theory, 2nd edition, 2001. Sections 1.1.1 and 1.1.2.