Steiner Forest Problem

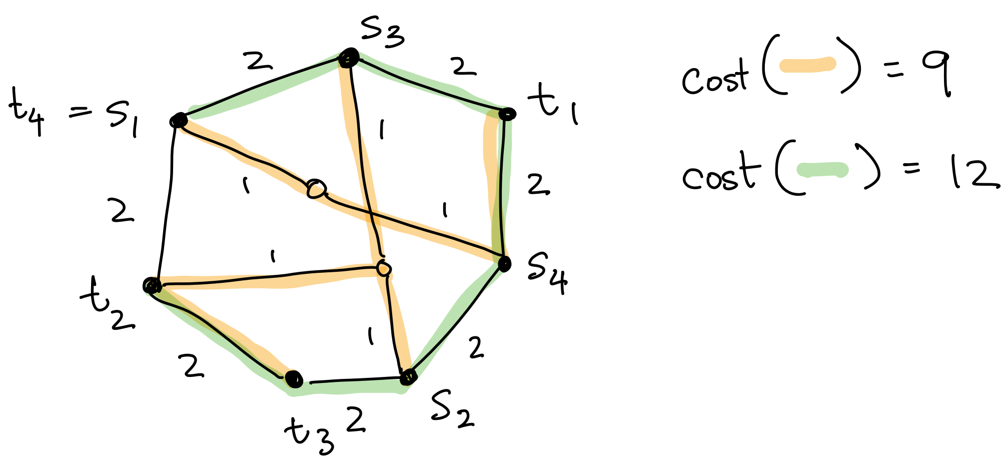
Input :
* Undirected Graph $G = (V, E)$
  Costs $c(e)$ on $e \in E$
* Terminal pairs : $\{(s_i, t_i)\}_{i=1 \text{ to } k}$

Output : $H \subseteq G$ a subgraph of G s.t

i) $\exists$ a path from $s_i \rightsquigarrow t_i$ in H

ii) $\sum_{e \in H} c(e)$ is minimized.

Example :



The problem generalizes

i) Shortest path $(k=1)$

ii) Spanning tree (all $\binom{n}{2}$ pairs)

iii) Steiner tree (all $\binom{R}{2}$ pairs)

LP relaxation

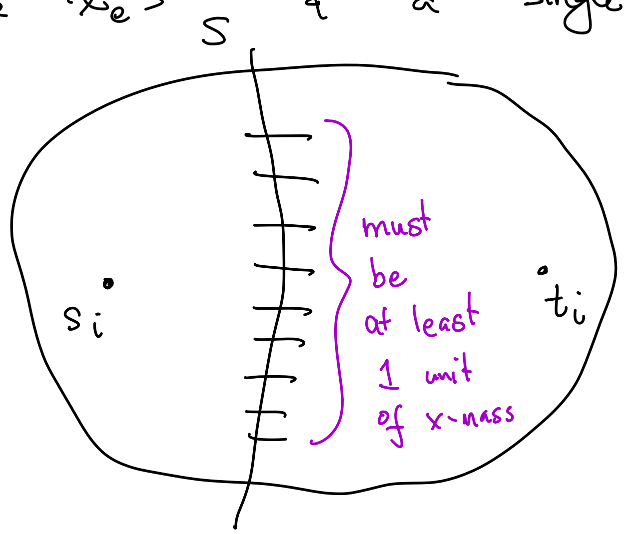
Variable: $x_e \quad \forall e \in E$ ind. whether we pick it or not

Constraint?:

* One can write a succinct LP capturing the fact that with "capacities" x_e on edges there is a "flow" of ≥ 1 unit from s_i to t_i, for each i

* However, one point of this lecture is to show that one needn't be too scared of LOTs of constraints

* Consider the x_e's & a single s_i, t_i pair



• Call a subset $S \subseteq V$ "binding" if $|\{s_i, t_i\} \cap S| = 1$ for some $1 \leq i \leq k$

$$\text{lp} := \min \sum_{e \in E} c(e)\, x_e$$

$$\underbrace{\forall S \subseteq V \text{ binding}}_{\mathcal{B}} : \quad x(\partial S) = \sum_{e \in \partial S} x_e \geq 1$$

$$x_e \geq 0$$

$$\text{dual} := \max \sum_{S \in \mathcal{B}} y_S$$

$$\forall e \in E : \quad \sum_{S \in \mathcal{B} : e \in \partial S} y_S \leq c(e)$$

$$y_S \geq 0$$

Primal-Dual Algorithm

* $\mathcal{A}$:= a collection of binding sets

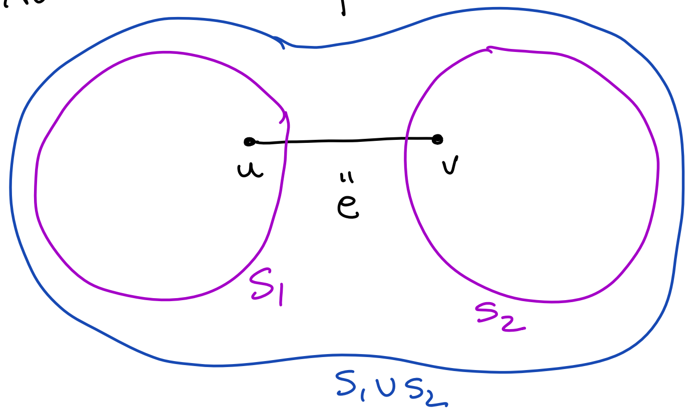
  Initally, $\mathcal{A} = \left\{ \{s_1\}, \{t_1\}, \{s_2\}, \{t_2\}, \ldots \right\}$

* We will maintain $\mathcal{A}$ is a collection of $\leq 2k$ disjoint sets & each terminal in some $A \in \mathcal{A}$

* $F \leftarrow \emptyset$ // primal forest

* Raise y_S $\forall S \in \mathcal{A}$ uniformly till for some $e \in E$ $\displaystyle\sum_{S : e \in \partial S} y_S = c_e$

* At that point, $F \leftarrow F \cup e$

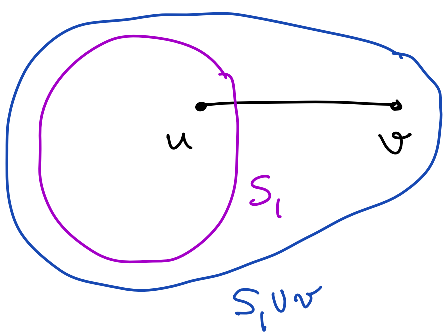
* 

→ · $\mathcal{A} \leftarrow \mathcal{A} \setminus S_1$

$\mathcal{A} \setminus S_2$

· if $S_1 \cup S_2 \in \mathcal{B}$

add $\mathcal{A} \leftarrow \mathcal{A} + S_1 \cup S_2$



→ $\mathcal{A} \leftarrow \mathcal{A} \setminus S_1$

$S_1 \cup v$ is binding

$\because v \notin T$

* At the end, we "prune" F.

We go over edges in reverse order they were added <u>deleting</u> any edge whose deletion keeps it a valid Steiner forest

Analysis

Observe

- The dual growth occurs in T rounds
- In round $t \in 1, \dots, T$,
  - $\mathcal{A}_t :=$ set of active binding sets
    - are <u>disjoint</u>
    - & increase by some $\Delta_t \geq 0$

$$\boxed{\bullet \ \text{dual} = \sum_{t=1}^{T} \sum_{S \in \mathcal{A}_t} \Delta_t = \sum_{t=1}^{T} \Delta_t \cdot |\mathcal{A}_t|}$$

- For every edge $e \in F$

$$c(e) = \sum_{S \in \mathcal{B} : e \in \partial S} y_S$$

$$= \sum_{t=1}^{T} \sum_{S \in \mathcal{A}_t : e \in \partial S} \Delta_t$$

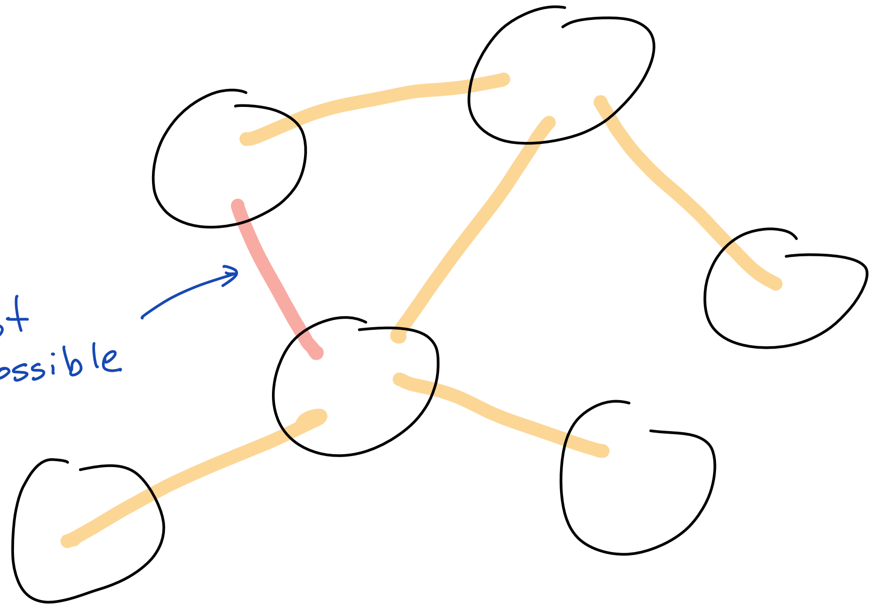
$$\therefore \ \text{cost}(F) = \sum_{e \in F} \sum_{t=1}^{T} \sum_{S \in \mathcal{A}_t : e \in \partial S} \Delta_t$$

$$= \sum_{t=1}^{T} \sum_{S \in \mathcal{A}_t} \Delta_t \cdot |\partial S \cap F|$$

Snapshot @ some t

⋮

A_t



not possible

MAIN CLAIM: F doesn't contain cycles in graph where vertices are A_t & (S, T) is an edge if $\exists\, e \in F$ s.t. $e \in S$ one pt, one in T

Pf:
- Look at the last edge added in F
- "Red" edge say. Well when we prune F, this will be deleted since $S \in A_t$ are connected.

CLAIM $\Rightarrow$ $$\sum_{S \in A_t} \deg_F(S) \leq 2(|A_t| - 1)$$
$$< 2|A_t|$$

$$\therefore \quad \text{cost}(F) < \underbrace{2 \sum_{t=1}^{T} \Delta_t \cdot |A_t|}_{= \sum_{S \in \mathcal{F}} y_S}$$

$\blacksquare$