EC 1.4 Descents

Def Let $\pi \in S_n$, $1 \le i \le n-1$.

i is a <u>descent</u> (ascent) of π if $\pi_i > \pi_{i+1}$ ($\pi_i < \pi_{i+1}$)

$\mathrm{Des}(\pi) = \{i : \pi_i > \pi_{i+1}\}$ <u>descent set</u> of π

$\mathrm{des}(\pi) = |\mathrm{Des}(\pi)|$

Ex $\mathrm{Des}(3\,4\,1\,7\,6\,2\,5) = \{2,4,5\}$ $\qquad \mathrm{des}(3417625) = 3$

Def Eulerian polynomial

$$A_n(x) = \sum_{\pi \in S_n} x^{1+\mathrm{des}(\pi)} = \sum_{k=1}^{n} \underbrace{\#\{\pi \in S_n \mid \mathrm{des}(\pi) = k-1\}}_{A(n,k)} x^k$$

$A(n,k)$ Eulerian number

Ex $A_3(x) = x + 4x^2 + x^3$

$A_4(x) = x + 11x^2 + 11x^3 + x^4$

$n \backslash k$	1	2	3	4	5
1	1				
2	1	1			
3	1	4	1		
4	1	11	11	1	
5	1	26	66	26	1

The rows are symmetric because

$$\mathrm{des}(\pi_n \cdots \pi_2 \pi_1) = n - 1 - \mathrm{des}(\pi_1 \pi_2 \cdots \pi_n)$$

Note also that, in each row, the numbers increase and then decrease, i.e., $A_n(x)$ is unimodal $\longrightarrow$ Possible final presentation.

Prop $A(n,k) = k\,A(n-1,k) + (n-k+1)\,A(n-1,k-1)$

Pf To obtain a permutation in S_n with $k-1$ descents, we insert n in a permutation $\pi \in S_{n-1}$ as follows:

- if π has $k-1$ descents, insert n at any descent or at the end
  $\to$ k possibilities
- if π has $k-2$ descents, insert n at any ascent or at the beginning
  $\to (n-2-(k-2))+1 = n-k+1$ possibilities

$\square$

Recall Foata's Fundamental transformation:

$$\pi = (a_1, a_2, \dots, a_{i_1})(a_{i_1+1}, \dots a_{i_2}) \cdots (a_{i_k+1}, \dots, a_n)$$

$a_1, a_{i_1+1}, a_{i_k+1}$: largest elements of their cycles, in ascending order

$$\hat{\pi} = a_1 a_2 \cdots a_{i_1} a_{i_1+1} \cdots a_n$$

Ex $\pi = (6)(75)(8241)(73) \mapsto \hat{\pi} = 675824173$

Then $\quad \pi(a_i) \geq a_i \quad$ iff $\begin{cases} a_i < a_{i+1} \\ \text{or } i = n \end{cases}$

i.e., i is not a descent of $\hat{\pi}$

So,
$$n - \mathrm{des}(\hat{\pi}) = \#\{ j \in [n] \mid \pi(j) \geq j \} = \mathrm{wexc}(\pi)$$

Def j is an excedance of π if $\pi(j) > j$ $\to$ $\mathrm{exc}(\pi) = \#$ excedances of π

j is a weak excedance of π if $\pi(j) \geq j$ $\to$ $\mathrm{wexc}(\pi) = \#$ weak excedances of π

$$= \mathrm{des}(\hat{\pi}_n \hat{\pi}_{n-1} \cdots \hat{\pi}_1) + 1 \qquad\qquad = n - \mathrm{exc}(\pi^{-1})$$

This proves the following.

Prop (MacMahon 1905)

$$A(n,k+1) = \#\{\pi\in S_n \mid des(\pi)=k\} = \#\{\pi\in S_n \mid exc(\pi)=k\}$$
$$= \#\{\pi\in S_n \mid wexc(\pi)=k+1\}$$ $\square$

Euler, in 1707, was interested in the series:

$$\sum_{i\geq 0} x^i = \frac{1}{1-x} \qquad \sum_{i\geq 0} i^2 x^i = \frac{x+x^2}{(1-x)^3}$$

differentiate and multiply by x

$$\sum_{i\geq 0} i x^i = \frac{x}{(1-x)^2} \qquad \sum_{i\geq 0} i^3 x^i = \frac{x+4x^2+x^3}{(1-x)^4} \quad \cdots$$

Prop
$$\sum_{i\geq 0} i^d x^i = \frac{A_d(x)}{(1-x)^{d+1}}$$

Pf By induction on d:

$\underline{d=0}$ ✓

$\underline{d-1\to d}$ Assume $\displaystyle\sum_{i\geq 0} i^{d-1}x^i = \frac{A_{d-1}(x)}{(1-x)^d}$

Differentiate both sides and multiply by x:

$$\sum_{i\geq 0} i^d x^i = x\,\frac{A_{d-1}'(x)(1-x)^d + A_{d-1}(x)\,d\,(1-x)^{d-1}}{(1-x)^{2d}}$$
$$= \frac{x(1-x)A_{d-1}'(x) + d\,x\,A_{d-1}(x)}{(1-x)^{d+1}}$$

Need to show: $A_d(x) \overset{?}{=} x(1-x)A_{d-1}'(x) + d\,x\,A_{d-1}(x)$
$$= xA_{d-1}'(x) - x^2A_{d-1}'(x) + d\,x\,A_{d-1}(x)$$

Recall

$A_{d-1}(x) = \sum A(d-1,k)x^k$

$A'_{d-1}(x) = \sum k A(d-1,k)x^{k-1}$

Take the coefficient of x^k on both sides:

$$A(d,k) \overset{?}{=} kA(d-1,k) - (k-1)A(d-1,k-1) + dA(d-1,k-1)$$

$$= kA(d-1,k) + (d-k+1)A(d-1,k-1),$$

which was proved earlier. $\square$

Prop

$$\sum_{d\geq 0} A_d(x)\frac{z^d}{d!} = \frac{1-x}{1-xe^{(1-x)z}}$$

Pf

$$\sum_{d\geq 0} A_d(x)\frac{z^d}{d!} = \sum_{d\geq 0}(1-x)^{d+1}\sum_{i\geq 0} i^d x^i \frac{z^d}{d!}$$

previous Prop

$$= (1-x)\sum_{i\geq 0} x^i \sum_{d\geq 0}\frac{i^d(1-x)^d z^d}{d!} = (1-x)\sum_{i\geq 0} x^i e^{i(1-x)z}$$

$$= \frac{1-x}{1-xe^{(1-x)z}}$$

$\square$

Def Major index of π: $\quad \mathrm{maj}(\pi) = \sum_{i \in \mathrm{Des}(\pi)} i$

Introduced by McMahon, named "major" by Foata because McMahon was a major in the British army.

Ex $\mathrm{maj}(4_{\uparrow}2_{\uparrow}1\,6\,7_{\uparrow}3\,5) = 1+2+5 = 8$

Thm $\#\{\pi \in S_n \mid \mathrm{inv}(\pi) = k\} = \#\{\pi \in S_n \mid \mathrm{maj}(\pi) = k\}$

Ex

$\pi \in S_3$	$\mathrm{maj}(\pi)$	$\mathrm{inv}(\pi)$
123	0	0
132	2	1
213	1	1
231	2	2
312	1	2
321	3	3

Equivalent formulation:

$$\sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)} = \sum_{\pi \in S_n} q^{\mathrm{maj}(\pi)}$$

For $n=3$, we get

$$1 + 2q + 2q^2 + q^3$$

Pf We construct a bijection

$$\varphi : S_n \longrightarrow S_n$$

such that $\mathrm{maj}(\pi) = \mathrm{inv}(\varphi(\pi))$

Build a sequence $\gamma_1, \gamma_2, \ldots, \gamma_n$, where $\gamma_1 = \pi_1$.

To get γ_{k+1} from γ_k:

- if $\pi_k < \pi_{k+1}$, split γ_k after each element $< \pi_{k+1}$
- if $\pi_k > \pi_{k+1}$, split γ_k after each element $> \pi_{k+1}$

Rotate each block cyclically to the right (i.e., move the last entry to the beginning), then add π_{k+1} at the end.

Finally, let $\varphi(\pi) = \gamma_n$.

Ex

$\pi = 6\,8\cdot 3\,9\cdot 4\cdot 1\,7\cdot 2\,5$ $\qquad$ $maj(\pi) = 2+4+5+7 = 18$

$\gamma_1 = 6$

$\gamma_2 = 6|8|$ split after each element > 3 and rotate

$\gamma_3 = 6|8|3|$ split after each element < 9 and rotate

$\gamma_4 = 6|8|39|$ - - - - - - > 4 and rotate

$\gamma_5 = 6|8|9|3|4|$ - - - - - > 1 - - - -

$\gamma_6 = 6|893|4|1|$ - - - - - < 7 - - - -

$\gamma_7 = 6|3|8|9|4|17|$ - - - - - > 2 - - - -

$\gamma_8 = 63|894|71|2|$ - - - - < 5 - - -

$\varphi(\pi) = \gamma_9 = 3\,6\,4\,8\,9\,1\,7\,2\,5$

$inv(\varphi(\pi)) = 5+6+0+1+4+0+2+0+0 = 18$

Claim: At each stage, $inv(\gamma_k) = maj(\pi_1\pi_2\cdots\pi_k)$

We prove the claim by induction on k:

$k=1$) $inv(\gamma_1) = 0 = maj(\pi_1)$

Suppose $\mathrm{inv}(\gamma_k) = \mathrm{maj}(\pi_1 \cdots \pi_k)$

Case $\pi_k > \pi_{k+1}$ | $k \in \mathrm{Des}(\pi)$, so $\mathrm{maj}(\pi_1 \cdots \pi_{k+1}) = \mathrm{maj}(\pi_1 \cdots \pi_k) + k$

Want to show $\mathrm{inv}(\gamma_{k+1}) = \mathrm{inv}(\gamma_k) + k$

In each compartment C of γ_k

$$|\underbrace{\_\ \_\ \_\ \_\ \_}_{<\pi_{k+1}}\ \underset{>\pi_{k+1}}{\_}\,|$$

- rotating the compartment creates $\mathrm{size}(C)-1$ new inversions
- appending π_{k+1} creates one inversion with each compartment

$\Rightarrow$ # inversions increases by $\displaystyle\sum_{\text{compartments } C} (\mathrm{size}(C)-1)$ + # compartments $= k$

Case $\pi_k < \pi_{k+1}$ | In this case, $\mathrm{maj}(\pi_1 \cdots \pi_{k+1}) = \mathrm{maj}(\pi_1 \cdots \pi_k)$

Want to show $\mathrm{inv}(\gamma_{k+1}) = \mathrm{inv}(\gamma_k)$

In each compartment C of γ_k

$$|\underbrace{\_\ \_\ \_\ \_\ \_}_{>\pi_{k+1}}\ \underset{<\pi_{k+1}}{\_}\,|$$

- rotating the compartment removes $\mathrm{size}(C)-1$ inversions
- appending π_{k+1} creates $\mathrm{size}(C)-1$ inversions with each compartment

$\Rightarrow$ # inversions does not change.

Finally, Ψ is a bijection because each of its steps can be reversed.

Note that to know where to split γ_{k+1} to recover γ_k, one needs to compare the *first* entry of γ_{k+1} with its last, and then split *before* each entry larger/smaller than the last entry accordingly. □

In fact, maj and inv satisfy a stronger property: the *joint* distribution of (inv, maj) is symmetric:

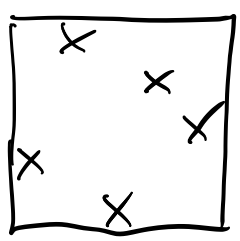
Thm
$$\sum_{\pi\in S_n} q^{\mathrm{inv}(\pi)}\, t^{\mathrm{maj}(\pi)} = \sum_{\pi\in S_n} q^{\mathrm{maj}(\pi)}\, t^{\mathrm{inv}(\pi)}$$

Ex For $n=3$, we get
$$1+qt+q^2t+qt^2+q^2t^2+q^3t^3$$

Unfortunately, it is not true that $\mathrm{maj}(\varphi(\pi)) = \mathrm{inv}(\pi)$ in general, so the above bijection doesn't prove this.

Def $\mathrm{IDes}(\pi) = \mathrm{Des}(\pi^{-1}) = \{ j : j+1 \text{ is to the left of } j \text{ in } \pi_1\pi_2\cdots\pi_n \}$

Ex $\pi = 25143$ $\quad \mathrm{IDes}(\pi) = \{1,3,4\}$



Obs $\pi^{-1}(j)$ is the position of j in $\pi_1\pi_2\cdots\pi_n$

Def $\mathrm{imaj}(\pi) = \mathrm{maj}(\pi^{-1}) = \sum_{j\in \mathrm{IDes}(\pi)} j$

Prop The above bijection $\varphi : \pi \to \tilde{\pi}$ preserves IDes, i.e.,

$$\mathrm{IDes}(\tilde{\pi}) = \mathrm{IDes}(\pi)$$

Pf We need to check that the relative order of j and $j+1$ never changes with the cyclic shifts that take γ_k to γ_{k+1}

Case $\pi_k > \pi_{k+1}$ Split after entries $> \pi_{k+1}$

$| s\, s\, s\, b |$ $\quad$ s stands for "small", i.e, $< \pi_{k+1}$

b - - - - "big" - - - $> \pi_{k+1}$

j & $j+1$ must be both small or both big, so their relative order does not change.

Case $\pi_k < \pi_{k+1}$ Same argument ☐

$$\widetilde{\pi^{-1}}^{-1} \xleftarrow{-1} \widetilde{\pi^{-1}} \xleftarrow{\varphi} \pi^{-1} \xrightarrow{-1} \pi \xrightarrow{\varphi} \tilde{\pi} \xrightarrow[-1]{\text{(take inverse)}} \tilde{\pi}^{-1}$$

$\widetilde{\pi^{-1}}^{-1}$	$\widetilde{\pi^{-1}}$	π^{-1}	π	$\tilde{\pi}$	$\tilde{\pi}^{-1}$
$\boxed{\mathrm{maj}}$	imaj	imaj	maj	inv	$\boxed{\mathrm{inv}}$
?	?	Des	IDes	IDes	Des
$\boxed{\mathrm{inv}}$	inv	maj	imaj	imaj	$\boxed{\mathrm{maj}}$

Let $\psi(\sigma) = \varphi(\varphi^{-1}(\sigma^{-1})^{-1})^{-1}$. The bijection $\sigma \mapsto \psi(\sigma)$ has the property that

$$\mathrm{inv}(\psi(\sigma)) = \mathrm{maj}(\sigma)$$
$$\mathrm{maj}(\psi(\sigma)) = \mathrm{inv}(\sigma).$$