Math 68 - Algebraic Combinatorics
Matroids

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Matroids are structures that generalize vector spaces, in the way that they are (finite) sets with a notion of independence.

## Definition

A (finite) matroid is a pair $(S, \mathcal{J})$, where $S$ is a finite set and $\mathcal{J}$ is a collection of subsets of $S$, satisfying:

(i) $\mathcal{J}$ is nonempty, and if $J \in \mathcal{J}$ and $I \subseteq J$, then $I \in \mathcal{J}$
(order ideal property, or simplicial complex property)

(ii) If $I, J \in \mathcal{J}$ and $|I| < |J|$, there exists $x \in J \setminus I$ such that $I \cup \{x\} \in \mathcal{J}$.

The elements of $\mathcal{J}$ are the independent sets.

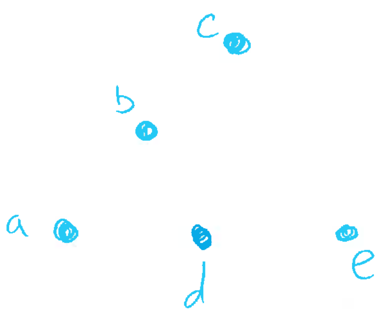
## Examples

- Vector spaces
  $S$ = a set of vectors (might not all be linearly independant).
  $\mathcal{J}$ = subsets of $S$ containing linearly independant vectors
- Hyperplane arrangements
  Taking $S$ as the vectors normal to the hyperplanes, and their span is a vector space.
  Moreover, if the arrangement is central, the poset we get by inclusion is isomorphic to the intersection lattice*

*It might not be obvious

## Example

Let $a, b, c, d, e$ be points in $\mathbb{R}^2$ and consider $\{x, y\}$ to be the minimal affine subspace containing $x$ and $y$ (if the set has two elements, it will always be a line, but $\{x, y, z\}$ could either be a line or a plane).




$S = \{a, b, c, d, e\}$

$J \in \mathcal{J}$ if the minimal affine subspace spanned by $J$ has dimension $\# J - 1$ ($\emptyset \in \mathcal{J}$)

The poset of minimal affine subspaces of $\mathbb{R}^2$ containing points in $S$: (order is inclusion).

$\square$ are the independent sets

for example, $abc \notin \mathcal{J}$ because removing one point among $a, b, c$ returns either $ab$, $bc$ or $ac$, and none of them are in $S$.

Note: All subsets of $S$ are part of the matroid, even if they do not appear in the poset above (which is called the "poset of flats").

Characterization of independence (for this poset):

$A \subseteq \{a, b, c, d, e\}$ is independent iff its rank in the poset is the cardinality of $A$.

## Example

$\mathcal{B}_3$ as a hyperplane arrangement

# Definitions

- A <u>basis</u> is a maximal independent set
- A <u>circuit</u> is a minimal dependent set.
- The <u>rank</u> of $A \subseteq S$ is $\max\{|X| : X \subseteq A, X \in \mathcal{I}\}$.

## Example

- For vector spaces, a basis is the usual thing we know.
- For a vector space, the rank is the dimension of the space it spans.

## Example

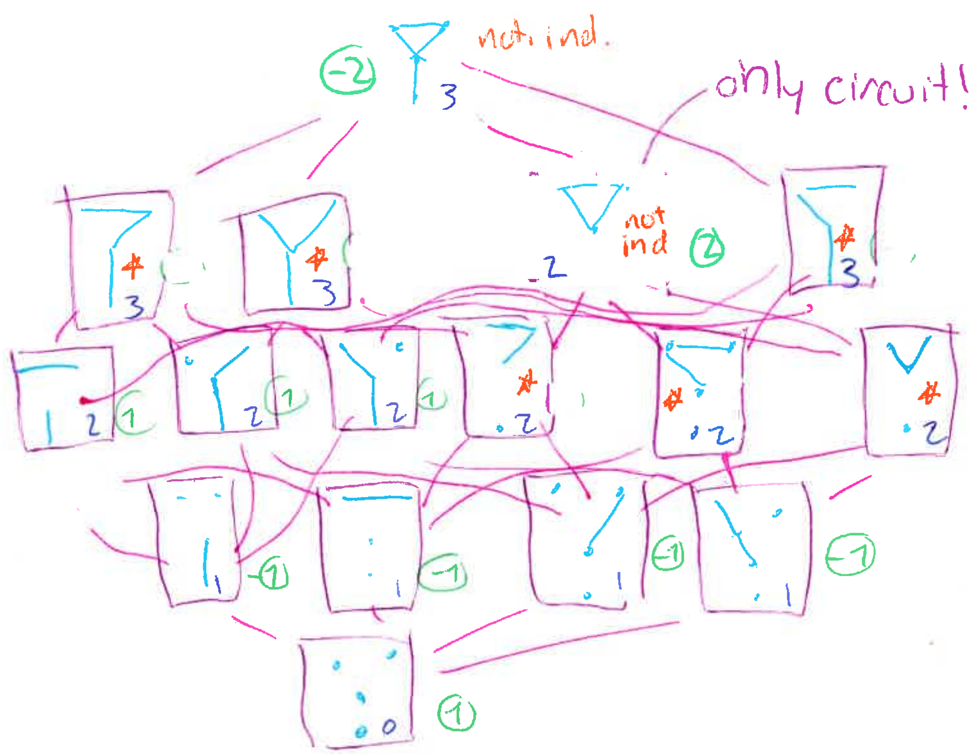
Matroids from a graph: $G = (V, E) =$ 

$E$ is the set of elements. A subset of edges is independent if they do not contain a cycle (or, if they are a forest).

$F = \{B \subseteq E \mid B \text{ is a forest}\}$

$(E, F)$ is a matroid

F
rank
$\mu$ (for a subposet)
$*$ not closed



## Definition

Fix a total order $\lhd$ on $S$.

A broken circuit is a subset of $S$ of the form $\hat{C} = C \setminus \{\min_{\lhd}(C)\}$, where $C$ is a circuit

## Example

For the graph on page 3, fix the following order

$$| \lhd \overline{\phantom{x}} \lhd / \lhd \backslash$$

There is only one circuit in $Z^E$: $C = \nabla$, and its minimal element is $\overline{\phantom{x}}$. Thus, the only broken circuit of $(E, F)$ is $\vee$.

## Example

Fix $a \lhd b \lhd c \lhd d \lhd e$. $M = (S, \mathcal{I})$ (on page 2) contains circuits $abc$, $ade$ and $bcde$. The 3 broken circuits of $M$ are $bc$, $de$ and $cde$.

The broken circuit complex $BC(M)$ is an abstract simplicial complex (i.e. $T \in BC(M)$ and $U \subseteq T \Rightarrow U \in BC(M)$) defined by

$$BC(M) = \{T \subseteq S : T \text{ contains no broken circuit}\}$$

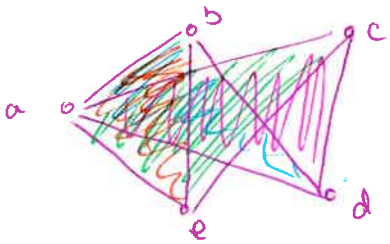
## Example

The broken circuit complex of $(E, F)$ is all the subsets of $E$ that do not contain $\vee$. As a simplicial complex, it is spanned by $\urcorner$ and $\nearrow$ (i.e. all elements of $BC$ are included in these two) and we get the following simplicial complex.

## Example

For the affine spaces on page 2, the broken circuits are bc, de and cde. And all elements of M that do not contain bc, de nor cde are included in either abd, abe, acd or ace.

Thus, the simplicial complex $BC(M)$ is




Dimension faces.

## Definition

The closure of a subset A of S for a matroid $(S,\mathcal{I})$ is

$$clos(A) = \{x \in S \mid rank(A \cup \{x\}) = rank(A)\}.$$

A subset A is closed if $clos(A) = A$.
Alternatively, A is closed if it is maximal among elements of the same rank.
e.g. in $(E,F)$, $\top$ is not closed, because $\nabla$ has same rank.

## Definition / Theorem

The characteristic polynomial of a matroid $M=(S,\mathcal{I})$ admits the equivalent definitions

- $$\chi_M(t) = \sum_{x \in Clos(M)} \mu(\hat{0}, x)\, t^{\underbrace{rank(M)}_{\text{max. rank for element in M.}} - rank(x)},$$
  where Clos is the induced subposet of closed subsets of of S, and the Möbius function is computed on that poset.
- $$\chi_M(t) = \sum_{i=0}^{rank(M)} (-1)^i f_{i-1}(BC(M))\, t^{rank(M)-i},$$
  where $f_{i-1}$ is the number of faces of dimension i in $BC(M)$.

## Examples

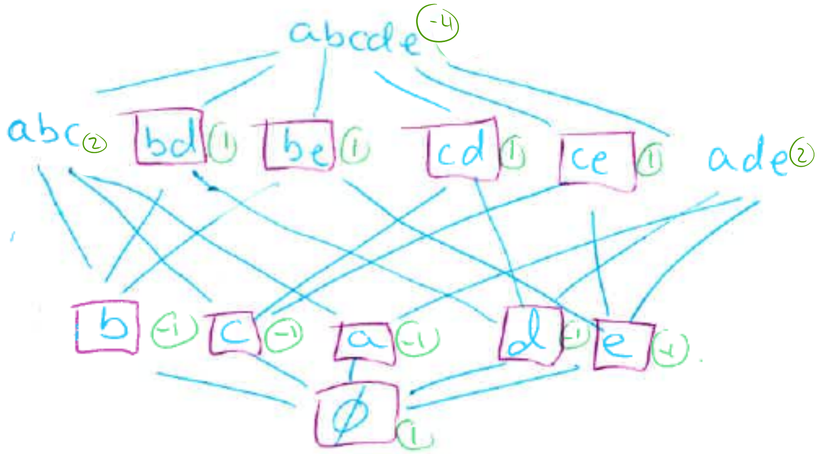
We can compute in two ways the characteristic polynomials of $M=(S,\mathcal{I})$ and $(E,F)$.

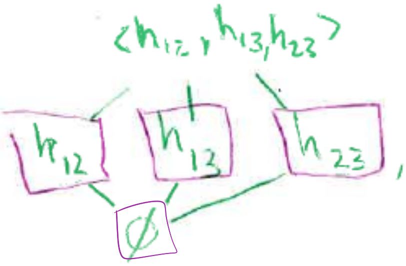
$$\chi_M(t) = t^3 - 5t^2 + 8t - 4 \quad \text{and} \quad \chi_{(E,F)}(t) = t^3 - 4t^2 + 5t - 2$$

References:

Richard P. STANLEY. An Introduction to Hyperplane Arrangements. §3,4.

## Page 2




where $h_{ij}$ is normal to $H_{ij}$.

## Page 4

| Dimension | # faces |
| --- | --- |
| -1 | 1 |
| 0 | 4 |
| 1 | 5 |
| 2 | 2 |

## Page 5

| Dimension | # faces |
| --- | --- |
| -1 | 1 |
| 0 | 5 |
| 1 | 8 |
| 2 | 4 |