# CS30 (Discrete Math in CS), Summer 2021 : Lecture 18

Topic: Probability: Variance

*Disclaimer: These notes have not gone through scrutiny and in all probability contain errors. Please discuss in Piazza/email errors to deeparnab@dartmouth.edu*

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- **Variance and Standard Deviation.**

  The expectation of a random variable is some sort of an "average behavior" of a random variable. However, the true value of a random variable may be no where close to the expectation. For instance, consider a random variable which takes the value $10000$ with probability $1/2$, and $-10000$ with probability $1/2$. What is $\mathbf{Exp}[X]$? Yes, it is $0$. Thus, there is significant *deviation* of $X$ from its expectation.

  The variance and standard deviation try to capture this deviation. In particular, the variance of a random variable is the *expected value of the square of the deviation.*

  Let $X$ be a random variable. The variance of $X$ is defined to be

  $$\mathbf{Var}[X] := \mathbf{Exp}\left[(X - \mathbf{Exp}[X])^2\right]$$

  That is, if we define another random variable $D := (X - \mathbf{Exp}[X])^2$, then $\mathbf{Var}[X]$ is the expected value of this new deviation random variable $D$.

  The *standard deviation* $\sigma(X)$ is defined to be $\sqrt{\mathbf{Var}(X)}$.

**Theorem 1.** $\mathbf{Var}[X] = \mathbf{Exp}[X^2] - (\mathbf{Exp}[X])^2$.

*Proof.*

$$\mathbf{Var}[X] = \mathbf{Exp}[(X - \mathbf{Exp}[X])^2] = \mathbf{Exp}[X^2 - 2X\mathbf{Exp}[X] + (\mathbf{Exp}[X])^2]$$

Then, we apply linearity of expectation to get

$$\mathbf{Var}[X] = \mathbf{Exp}[X^2] - 2\mathbf{Exp}[X] \cdot \mathbf{Exp}[X] + (\mathbf{Exp}[X])^2 = \mathbf{Exp}[X^2] - (\mathbf{Exp}[X])^2$$

□

A useful corollary (something we observed in the last lecture notes):

**Theorem 2.** For any random variable $\mathbf{Exp}[X^2] \geq (\mathbf{Exp}[X])^2$.

*Proof.* $\mathbf{Var}[X]$ is the expected value of $(X - \mathbf{Exp}[X])^2$. That is, $\mathbf{Var}[X]$ is the expected value of a random variable which is always non-negative. In particular, $\mathbf{Var}[X]$ is non-negative. Which in turn means $\mathbf{Exp}[X^2] - (\mathbf{Exp}[X])^2 \geq 0$. Rearranging implies the corollary. □

Examples

– *Roll of a die.* Let $X$ be the roll of a fair 6-sided die. We know that $\mathbf{Exp}[X] = 3.5$. To calculate the variance, we can use the deviation $D := (X - \mathbf{Exp}[X])^2 = (X - 3.5)^2$. Usinhg this, we get

$$\mathbf{Var}[X] = \mathbf{Exp}[D] = \frac{1}{6}\left((2.5)^2 + (1.5)^2 + (0.5)^2 + (0.5)^2 + (1.5)^2 + (2.5)^2\right) = \frac{35}{12}$$

– *Toss of a biased coin.* Let $X$ be a Bernoulli random variable taking value 1 if a coin tosses heads, and 0 otherwise. Suppose the probability of heads was $p$. Recall, $\mathbf{Exp}[X] = p$. Also note since $X$ is a indicator random variable, $X^2 = X$. Thus, $\mathbf{Exp}[X^2] = p$ as well. We can calculate the variance as

$$\mathbf{Var}[X] = \mathbf{Exp}[X^2] - (\mathbf{Exp}[X])^2 = p - p^2 = p(1-p)$$

– *Indicator Random Variable.* Using the above toss of a biased coin example, we see that for any event $\mathcal{E}$, the variance of the indicator random variable is

$$\mathbf{Var}[\mathbf{1}_{\mathcal{E}}] = \mathbf{Pr}[\mathcal{E}] \cdot (1 - \mathbf{Pr}[\mathcal{E}]) = \mathbf{Pr}[\mathcal{E}] \cdot \mathbf{Pr}[\neg\mathcal{E}]$$

**Theorem 3.** If $X$ is a random variable, and $c$ is a "scalar" (a constant), then $Z = c \cdot X$ is another random variable. $\mathbf{Var}[c \cdot X] = c^2 \cdot \mathbf{Var}[X]$.

*Proof.*

$$\mathbf{Var}[c \cdot X] = \mathbf{Exp}[c^2X^2] - (\mathbf{Exp}[cX])^2 = c^2\mathbf{Exp}[X^2] - c^2\,(\mathbf{Exp}[X])^2 = c^2\mathbf{Var}[X]$$

□

The next theorem is a *linearity of variance* result for ***independent*** random variables.

**Theorem 4.** For any two ***independent*** random variables $X$ and $Y$, let $Z := X + Y$. Then,

$$\mathbf{Var}[Z] = \mathbf{Var}[X] + \mathbf{Var}[Y]$$

*Proof.* By definition of variance, we get

$$\mathbf{Var}[X+Y] = \mathbf{Exp}[(X+Y)^2] - (\mathbf{Exp}[X] + \mathbf{Exp}[Y])^2 \tag{1}$$

Now, we expand the first term in the RHS to get

$$\begin{aligned}
\mathbf{Exp}[(X+Y)^2] &= \mathbf{Exp}[X^2 + 2XY + Y^2] \\
&= \mathbf{Exp}[X^2] + 2\mathbf{Exp}[XY] + \mathbf{Exp}[Y^2] && \text{Linearity of Expectation} \\
&= \mathbf{Exp}[X^2] + 2\mathbf{Exp}[X]\mathbf{Exp}[Y] + \mathbf{Exp}[Y^2] && \text{Since } X \text{ and } Y \text{ are independent.}
\end{aligned} \tag{2}$$

Next, we expand the second term in the RHS of (1), to get

$$(\mathbf{Exp}[X] + \mathbf{Exp}[Y])^2 = (\mathbf{Exp}[X])^2 + 2\mathbf{Exp}[X]\mathbf{Exp}[Y] + (\mathbf{Exp}[Y])^2 \tag{3}$$

Subtracting (3) from (2), we get

$$\begin{aligned}
\mathbf{Var}[X+Y] &= \left(\mathbf{Exp}[X^2] - (\mathbf{Exp}[X])^2\right) + \left(\mathbf{Exp}[Y^2] - (\mathbf{Exp}[Y])^2\right) \\
&= \mathbf{Var}[X] + \mathbf{Var}[Y]
\end{aligned} \tag{4}$$

□

We can generalize the above proof to many random variables. In particular, we can say that if $X_1, X_2, \ldots, X_k$ are mutually independent random variables, then the variance of the sum is the sum of the variances. However, we *don't need mutual independence*. Pairwise independence suffices. The proof is given as a solution to the UGP; perhaps you can try it. There is nothing more than the algebra above except there are $k$ things adding up.

**Theorem 5.** For any $k$ ***pairwise independent*** (and therefore also for mutually independent) random variables $X_1, X_2, \ldots, X_k$,

$$\mathbf{Var}\left[\sum_{i=1}^{k} X_i\right] = \sum_{i=1}^{k} \mathbf{Var}[X_i]$$