CS30 (Discrete Math in CS), Summer 2021 : Ungraded Practice Problems 3

Due: Not for Submission

Topics: Combinatorics

Product, Sum, and Division Principle

Problem 1. How many 5 digit numbers are there with all digits even? 0 is even and the 5-digit number cannot start with 0. ☕

Problem 2. You are given 10 books with different titles. 6 of them are large and 4 of them small. How many ways can you stack these books up such that a large book is never stacked over a small book? ☕☕

Problem 3. How many four letter (not necessarily dictionary) words can you make which have *at least* one vowel? A vowel is a letter from the set $\{a, e, i, o, u\}$ (so y is not a vowel.) ☕☕

Problem 4. (Number of functions)

a. ☕ How many functions $f : A \to B$ are there where $|A| = n$ and $|B| = m$?

b. ☕ How many *bijective* functions $f : A \to B$ are there where $|A| = n$ and $|B| = n$?

Problem 5. How many 5 digit numbers are there such that there is at least one digit less than or equal to 3, at least one digit from the set $\{4, 5, 6\}$, and at least one digit greater than or equal to 7. ☕☕

Problem 6. How many strings of length n can you form from the characters $\{A, C, G, T\}$ such that no two consecutive characters are same.☕

Problem 7. How many bit vectors of length 10 contain at least five consecutive 0s. ☕☕☕

Problem 8. How many ways can you arrange 5 different people in a circular table? Note that two arrangements which read the same clockwise are the same. ☕

Problem 9. Let A be a set of $n \geq 1$ elements. Let $a \in A$ be an arbitrary element. What is the number of subsets of A which contain a? ☕

Problem 10. In a certain population of m people, there are n groups. Each group contains exactly q people and every person is in exactly d groups. What is the relation between the numbers n, m, d, and q? ☕☕

The Four Fold Formula

Problem 11. ☕
How many non-negative integer solutions are there to the equation $a + b + c = 100$?

Problem 12. ☕☕
How many non-negative integer solutions are there to the equation $a + b + c \leq 100$?

Problem 13. ☕☕
How many n-bit strings are there with exactly k ones such that no two ones are adjacent? For example, if $n = 5$ and $k = 2$, then the allowed bitstrings are `00101`, `01001`, `01010`, `10001`, `10010`, `10100`.

Binomial Expansion and Combinatorial Identities

Problem 14. ☕☕
Given a subset U with $|U| = n$, what is the cardinality of the set

$$X := \{(A, B) : A \subseteq U, B \subseteq A\}$$

Problem 15. ☕☕
For any natural number n, prove that

$$\left(1 + \frac{1}{n}\right)^n < e$$

where $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \approx 2.7183$ is the base of the natural algorithm.

The following examples show how combinatorial identities are proven. They are tricky, since you need to figure out which set it is counting. It comes with practice. I suggest trying them hard, and then looking at the answers.

Problem 16. ☕☕
Prove the following identity combinatorially: for any naturals n and $k \leq n$, we have

$$k \cdot \binom{n}{k} = n \cdot \binom{n-1}{k-1}$$

Problem 17. (Vandermonde's Identity) ☕☕☕
For any naturals m, n, r,

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k} \cdot \binom{n}{r-k}$$

Problem 18. (Consequence of Vandermonde's Identity) ☕☕
Prove the following combinatorial identity:

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

Problem 19. (Hockey Stick Identity) ☕☕☕
For any natural numbers n and r with $n \geq r$.

$$\sum_{m=r}^{n} \binom{m}{r} = \binom{n+1}{r+1}$$

Bijective Proofs

Problem 20 (Counting Intersections)**.** (Problem 1 in Pset 0) ☕☕

Suppose you are given n pins on a circular integrated chip. All pairs of pins must be connected by straight line wires. Any time two wires are about to intersect, you have to put in a plastic micro-clip to prevent them from contacting. How many plastic micro-clips would you need? Give reasons for your answer. You may assume that you are unlucky enough that no three wires meet at the same point.

As an illustration, I am drawing the situation for $n = 5$ and 6 — I urge you to draw the $n = 7$ case too.

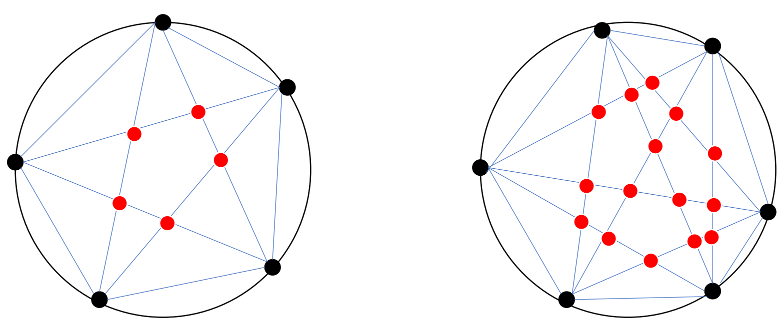

Figure 1: The $n = 5$ and 6 case for Problem 20. The red circles in the interior are the intersections. There are 5 intersections when $n = 5$ and 15 intersections when $n = 6$. Try finding the answer when $n = 7$. And then search for these numbers in Pascal's triangle.

Problem 21 (Counting Compositions)**.** ☕☕

Given a number n, a *composition* is a sequence $(a_1, a_2, \ldots)$ whose entries are *positive* integers which sum up to n. For example, for $n = 5$, the sequences (5), $(1, 4)$, $(4, 1)$, and $(2, 3)$ are four *different* sequences of the number 5.

How many sequences does the number n have? Again, I would urge you to find the actual answer by brute force for $n = 2, 3, 4$. The answer will emerge. And then you can try and prove it.