CS30 (Discrete Math in CS), Summer 2021

Drill 6+7

Topic: Induction

Instructions

- Please submit all homework electronically in PDF, ideally typeset using LaTeX. If your handwriting is not legible, you may get 0 points.
- The drills below are supposed to be quick to do and quick to check. If a grader cannot read and understand your solution to a given drill exercise in **1 minute**, you may get a 0.
- **Collaboration Policy:** You should be able to and indeed should do the drills on your own. Collaboration is not allowed. You can ask clarification questions on Ed Discussion **privately**; the instruction team may choose to make it public. You can refer to the recommended textbook, your own course notes, posted videos, and the posted lecture notes. **Not** the web. When in doubt, consult the instructor.

Exercise 1. (**4 points**)
Prove using induction:

$$\text{For all natural numbers } n, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Clearly write:

- The predicate $P(n)$.
- Establish the base case.
- In the inductive case, clearly state the inductive hypothesis, that is, what you are assuming is true, and clearly state what you want to prove.

Exercise 2. (**4 points**) The Fibonacci numbers are defined as $F_1 = 1, F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Prove that $F_n \leq 2^n$. Clearly write:

- The predicate $P(n)$.
- Establish the base case.
- In the inductive case, clearly state the inductive hypothesis, that is, what you are assuming is true, and clearly state what you want to prove.

Exercise 3. (**2 points**)
Find which line has a bug in the following "proof" of a blatantly wrong statement. Clearly state what is wrong.

Claim 1. Let S be any non-empty finite subset of $\mathbb{N}$. Then either all numbers in S are odd or all numbers in S are even.

Proof.

1. Let $P(n)$ be the predicate "Any subset $S \subseteq \mathbb{N}$ with $|S| = n$ either consist entirely of even numbers or consist entirely of odd numbers. We need to prove $\forall n \in \mathbb{N} : P(n)$. We proceed by induction.

2. We check the **base case**, that is, we assert $P(1)$ is true. Indeed, any subset $S \subseteq \mathbb{N}$ with $|S| = 1$ consists of a single number. This number is either odd or even. In the first case, all elements of S are odd; in the second case, all elements of S are even.

3. We proceed to the **inductive case**. Fix a natural number k and assume $P(k)$ is true. That is, we assume that for *any* subset $S \subseteq \mathbb{N}$ with $|S| = k$, either all its elements are odd, or all its elements are even. We need to prove $P(k+1)$ is true.

4. In order to do so, fix a set $S \subseteq \mathbb{N}$ such that $|S| = k+1$. Call its elements $\{a_1, a_2, \dots, a_{k+1}\}$. We need to show either all the a_i's are odd, or all the a_i's are even.

5. Consider the *subset* $T_1 \subseteq S$ defined as $T_1 = \{a_1, \dots, a_k\}$. Note $|T_1| = k$. Therefore, by the induction hypothesis that $P(k)$ is true, either all elements of T_1 are odd, or all elements of T_1 are even. Suppose they are all odd, that is, all the elements $a_1, \dots, a_k$ are odd. In particular, a_2 is odd.

6. Now consider the *subset* $T_2 \subseteq S$ defined as $\{a_2, \dots, a_{k+1}\}$. Note $|T_2| = k$. Therefore, by the induction hypothesis that $P(k)$ is true, either all elements of T_2 are odd, or all elements of T_2 are even. But $a_2 \in T_2$, and a_2 is odd. Therefore all the elements of T_2 are odd, and in particular a_{k+1} is odd.

7. Therefore, from Line 5 and Line 6, we conclude all the elements of S are odd. Indeed, if in Line 5, we would have assumed that all elements of T_1 were even, that would imply all the elements of S are even. In any case, all the elements of S are either even or odd. That is, we have established $P(k+1)$, and therefore, by induction, we have established $\forall n \in \mathbb{N} : P(n)$.

$\square$