CS30 (Discrete Math in CS), Summer 2021 : Lecture 4

Topic: Predicate Logic

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- **Predicates.** A *predicate* is a function $P(x)$ with a variable x (or multiple variables such as x, y) where (a) the variable takes a value from a certain set (say, integers for now), and (b) given a fixed value for the variable x, the predicate $P(x)$ becomes a proposition.

  Here is an example.
  $$P(x) = (x \text{ is an odd number})$$
  where the variable x takes a value among the positive integers. Given a fixed value for x, say $x = 3$, we get a proposition $P(3)$ which is true. On the other hand, $P(2)$ is false.

  As mentioned above, predicates can take multiple variables. For instance,
  $$P(x, y) = (x + y \text{ is an even number})$$
  is a predicate where both its variables take a value among the integers. $P(2, 3)$ is false, but $P(1, 9)$ is true.

  Exercise: *Define a predicate of your own.*

- **Quantifiers.** A predicate by itself is neither true nor false. The power of predicates arise when used with *quantifiers*. There are two quantifiers:
  $$\forall, \quad \text{which stands for "for all"} \qquad \text{and} \qquad \exists, \quad \text{which stands for "there exists"}$$
  Using this we can get new propositions which take true or false values. Formally,

  Remark: *Given any predicate $P(\cdot)$ and a set S such that $P(x)$ takes value* true *or* false *for any $x \in S$, the following are propositions in* predicate logic*:*
  $$\forall x \in S: \ P(x) \qquad \exists x \in S: \ P(x)$$
  The set S is called the domain of discourse*.*

  For example,
  $$\phi := \forall x \in \mathbb{Z} : P(x) \quad \text{where, } P(x) = (x \text{ is an odd number})$$
  is a statement which takes a value true or false. The set of integers $\mathbb{Z}$ is the *domain of discourse*. It is true if *for every* fixed $x \in \mathbb{Z}$, that is, every fixed integer x, the proposition $P(x)$ is true. As you can see, ϕ takes the value false (because not every integer is odd.)

  The following is also a proposition.
  $$\psi := \exists x \in \mathbb{Z} : P(x) \quad \text{where, } P(x) = (x \text{ is an odd number})$$
  It takes the value true if *for some* fixed $x \in \mathbb{Z}$, the proposition $P(x)$ is true. As you can see, ψ takes the value true (since some integer, say 3, is odd.)

- **Thinking of Quantifiers as ANDs and ORs.** Suppose $S = \{x_1, x_2, \ldots, x_k\}$ is a finite domain of discourse. Then, a way of thinking about quantifiers is

  $$\forall x \in S : P(x) \equiv P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_k) \qquad\qquad \exists x \in S : P(x) \equiv P(x_1) \vee P(x_2) \vee \cdots \vee P(x_k)$$

  Note that when the set S is infinite (such as natural numbers or integers), then the above is just a tool to help you think, and not formally correct.

- **Examples of statements expressed in Predicate Logic.** Below we show the expressiveness of predicate logic (quantifiers + predicates) using some examples of statements and they expressed in predicate logic. This helps formalize English statements.

  – *A number is divisible by* 4 *if and only if its last two digits are*.

  We define two predicates $P(n) = (n$ is divisible by $4)$ and $Q(n) = ($Last two digits of n are divisible by $4)$. The variables for both predicates takes value in natural numbers. The above statement expressed in predicate logic is

  $$\forall n \in \mathbb{N} : ((\neg P(n) \vee Q(n)) \wedge (\neg Q(n) \vee P(n)))$$

  – *An irrational number raised to power an irrational number can be a rational number.*

  We define a predicate $P(z) = (z$ is an irrational number$)$, which takes the variable in *real numbers*. The above statement can be written in predicate logic as

  $$\exists x \in \mathbb{R}, y \in \mathbb{R} : (P(x) \wedge P(y) \wedge \neg P(x^y))$$

  Exercise: *Write the following in predicate logic. Clearly define your predicates.*

  – *All prime numbers larger than or equal to* 3 *are odd.*

  – *Any perfect square is either odd or it has a* $0, 4,$ *or* 6 *in the units place.*

- **Caution!** We know that $\wedge$'s and $\vee$'s nicely distribute. That is, $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$. Taking cue from this, one may be inclined to think

  $$\forall x \in A : (P(x) \vee Q(x)) \quad \equiv ? \quad (\forall x \in A : P(x)) \vee (\forall x \in A : Q(x))$$

  That is, does the $\forall$ "distribute" over $\vee$. Do not make this mistake!

  The $\forall$ operator and the $\vee$ operator do not distribute

  Indeed, if you translate in English, the equivalence looks fishy. The left side says "for every x, either $P(x)$ occurs or $Q(x)$ occurs." The right side says "Either for every x, $P(x)$ occurs, or for every x, $Q(x)$ occurs." The second statement seems stronger (does it to you?).

  To show the non-equivalence, we just need to give an example where the LHS and RHS evaluate to two different things. You may want to pause here and figure it out yourself.

  Suppose $P(x)$ is the proposition (x is even) and $Q(x)$ is the proposition (x is odd), and the domain of discourse is $\mathbb{N}$. Then, the LHS formula evaluates to true: every natural number is either odd or even. The RHS formula is false. $\forall x \in \mathbb{N} : P(x)$ is false since all numbers are not even. $\forall x \in \mathbb{N} : Q(x)$ is false since all numbers are not odd. And $\mathsf{false} \vee \mathsf{false}$ is false.

  Similarly,

The $\exists$ operator and the $\wedge$ operator do not distribute

However, $\forall$'s do "distribute" over $\wedge$'s and $\exists$'s do "distribute" over $\vee$'s.

$$\forall x \in A : (P(x) \wedge Q(x)) \equiv (\forall x \in A : P(x)) \wedge (\forall x \in A : Q(x))$$

$$\exists x \in A : (P(x) \vee Q(x)) \equiv (\exists x \in A : P(x)) \vee (\exists x \in A : Q(x))$$

Do you see why this is? Hint: think of the quantifiers as a chain of $\wedge$'s and $\vee$'s, and use the associativity property of these operators.

- **Negations of statements in predicate logic.** Say you would like to prove a statement such as one of the examples given above. Suppose you decide to prove the statement by contradiction. The first step is to understand what the contradiction even means, that is, you need to figure out the negation of the statement you want to prove.

$$\neg (\forall x \in S : P(x)) \equiv \exists x \in S : \neg P(x) \qquad \text{(Negation of a } \forall\text{)}$$

$$\neg (\exists x \in S : P(x)) \equiv \forall x \in S : \neg P(x) \qquad \text{(Negation of a } \exists\text{)}$$

One way to see why, say (Negation of a $\forall$), is true, is to use the "propositional logic view" of predicate logic statements, and apply De Morgan's Law. More formally, suppose $S = \{x_1, \ldots, x_k\}$ is finite. Then,

$$\forall x \in S : P(x) \equiv P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_k)$$

and therefore,

$$\neg \forall x \in S : P(x) \equiv \quad \neg P(x_1) \vee \neg P(x_2) \vee \cdots \vee \neg P(x_k) \quad \equiv \exists x \in S : \neg P(x)$$

A (correct) proof of (Negation of a $\forall$) which doesn't deal with infinities is the following. We show whenever the formula $\neg(\forall x \in S : P(x))$ takes the value `true`, the formula $\exists x \in S : \neg P(x)$ takes the value `true`, and vice-versa.

The formula $\neg(\forall x \in S : P(x))$ takes value `true` implies the formula $\forall x \in S : P(x)$ takes the value `false`. If the formula $\forall x \in S : P(x)$ takes value `false`, there must exist an $a \in S$ such that $P(a)$ is `false` which implies $\neg P(a)$ is `true`, that is $\exists x \in S : \neg P(x)$ is `true`. The vice-versa direction, which is crucial, is left as an exercise to the reader.

Exercise: *Do the exercise.*

Exercise: *Prove* (Negation of a $\exists$)

This allows us to take say the negation of the statement "An irrational number raised to power an irrational number can be a rational number." In predicate logic, the negation is

$$\neg (\exists x \in \mathbb{R}, y \in \mathbb{R} : (P(x) \wedge P(y) \wedge \neg P(x^y))) \equiv \forall x \in \mathbb{R}, y \in \mathbb{R} : (\neg P(x) \vee \neg P(y) \vee P(x^y))$$

which we can read out in English as "For any two real numbers x, y, either x is rational or y is rational or x^y is irrational."

This may or may not be the converse of the statement given that you would've thought of (For me it was not). But let's rewrite it slightly to get

$$\begin{aligned}\forall x \in \mathbb{R}, y \in \mathbb{R} : (\neg P(x) \vee \neg P(y) \vee P(x^y)) &\equiv \forall x \in \mathbb{R}, y \in \mathbb{R} : (\neg (P(x) \wedge P(y)) \vee P(x^y)) \\ &\equiv \forall x \in \mathbb{R}, y \in \mathbb{R} : ((P(x) \wedge P(y)) \Rightarrow \neg P(x^y))\end{aligned}$$

where we have used that $p \Rightarrow q$ is equivalent to $\neg p \vee q$. Now, the English version of this statement is "For any two real numbers x, y, if both x and y are irrational, then x^y is irrational." This was the converse I would've thought of.

- **Nested Quantifiers.** In the example above regarding irrational numbers, we had two variables x, y and we used $\forall$ implicitly for both. However, there are some instances where we have $\forall$ for one variable and $\exists$ for the other. For example, consider the statement "For every integer x, there is an integer y which is bigger than it."

  In predicate logic we will write this as

  $$\phi := \forall x \in \mathbb{Z} \ \ \exists y \in \mathbb{Z} : \ P(x, y)$$

  where $P(x, y)$ is the predicate taking the value true if $y > x$. What is the truth value of this formula? It is true. A proof of this is like a game between you and an adversary. The adversary is the $\forall$ person who gives x. Once x is fixed, our job is to find a $y \in \mathbb{Z}$ such that $P(x, y)$ is true. $y = x + 1$ suffices.

  Note the **order** of quantifiers. If we flip this order, we get something completely different.

  $$\psi := \exists y \in \mathbb{Z} \ \ \forall x \in \mathbb{Z} : \ P(x, y)$$

  which in English translates to "There exists an integer y which is bigger than every integer x." The value of this statement is false. Once again, you can play the game with the adversary who claims this formula is true. To do so, she produces the integer y^* and claims $\forall x \in \mathbb{Z} : P(x, y^*)$ is true. But then you show $x = y^* + 1$ for which $P(x, y)$ is false.

- **Negations of Nested Quantifiers.** The negation of the formula

  $$\phi := \forall x \in \mathbb{Z} \ \ \exists y \in \mathbb{Z} : \ P(x, y)$$

  can be written using the above rules as

  $$\neg\phi \equiv \qquad \exists x \in \mathbb{Z} \neg (\exists y \in \mathbb{Z} : \ P(x, y)) \qquad \equiv \exists x \in \mathbb{Z} \ \ \forall y \in \mathbb{Z} : \neg P(x, y)$$

- **More examples of statements expressed in predicate logic.**

  - *Every non-zero real number has a reciprocal.*
    We need to show that for any real number x such that $x \neq 0$, there is some other real number y such that $xy = 1$. To do so, we define the predicate $P(x, y) = \mathsf{true}$, if $xy = 1 \quad = \mathsf{false}$, otherwise. The above statement is then expressed as

    $$\phi = \forall x \in \mathbb{R} : ((x \neq 0) \Rightarrow \exists y \in \mathbb{R} : P(x, y))$$

Note how one quantifier (the $\exists$) is inside the statement of another quantifier (the $\forall$). That is perfectly fine. One would ideally like to move all the quantifiers to the front, but that is not always possible. However, let's try for the above example.

Using Implication as OR, (and using $\neg(x \neq 0)$ is the same as $(x = 0)$)we get

$$\phi \equiv \forall x \in \mathbb{R} : ((x = 0) \vee \exists y \in \mathbb{R} : P(x, y))$$

We *cannot* distribute. So, this seems to be as simple as one can get.

One can make the formula "nicer" (having all quantifiers up front) by *changing the domain of discourse*. Indeed,

$$\phi \equiv \forall x \in \mathbb{R} \setminus \{0\} \exists y \in \mathbb{R} : P(x, y)$$

Do you see the equivalence?

Answers to Exercises.

- You define a predicate on your own. I have defined many above (and below)
- The domain of discourse is natural numbers $\mathbb{N} := \{1, 2, 3, \ldots\}$. $P(x)$ is true if x is prime. $O(x)$ is true if x is odd. Then, the statement "all prime numbers larger than or equal to 3 are odd" is written as
$$\forall x \in \mathbb{N} : \Big((x \geq 3) \wedge P(x)\Big) \Rightarrow O(x)$$
  For the second statement, the domain is still naturals. $S(x)$ is a predicate taking the value true if x is a perfect square. $O(x)$ is as before. Let's define $E(x)$ be the predicate which is true if x's units place has $0, 4$ or 6. Then,
$$\forall x \in \mathbb{N} : S(x) \Rightarrow \Big(O(x) \vee E(x)\Big)$$
  There are of course many equivalent ways of writing the above.
- The formula $\exists x \in S : \neg P(x)$ takes value true implies that there exists some $x \in S$ such that $\neg P(x)$ is true. Let this be a. And thus $\neg P(a)$ is true, which implies $P(a)$ is false. But this implies $(\forall x \in S : P(x))$ is false. This, in turn, implies $\neg(\forall x \in S : P(x))$ is indeed true. Done.
- The proof of
$$\neg(\exists x \in S : P(x)) \equiv \forall x \in S : \neg P(x) \qquad \text{(Negation of a } \exists\text{)}$$
  is very similar to the "$\forall$" case.

  If the LHS evaluates to true, then $(\exists x \in S : P(x))$ evaluates to false, which implies there doesn't exist any $x \in S$ with $P(x) = \mathsf{true}$. That is, for every $x \in S$, we have $\neg P(x)$ is true. Which means $\forall x \in S : \neg P(x)$ is true.

  On the other hand, if $\forall x \in S : \neg P(x)$ is true, then $P(x) = \mathsf{false}$ for all $x \in S$. Which implies $\exists x \in S : P(x)$ is indeed false. And thus the LHS is true.