Math 3 Winter 2018

Final Exam

March 9, 2018

Your name (please print): ______________________________

Circle your section:

Pauls (9L)   Ratti (10)   Malik (11)   Malik (12)

Instructions:

- This exam is closed book and closed notes. **No calculators, phones, or other devices may be used during the exam.**
- **The Honor Principle requires that you may not give or receive any help during the exam, though you may ask instructors for clarification.**
- This exam consists of **15** questions and you have **three (3)** hours to complete them. Point values are indicated for each question.
  - For multiple choice questions, no justification is required.
  - For long answer questions, you must justify all of your answers completely and show your work to receive full credit. Write clearly in complete sentences. Partial credit may be assigned to incomplete solutions.
- Please do all your work in this exam booklet. There are five (5) pages for extra work at the end of the booklet. **No work on the extra pages will be considered unless you specifically reference it on the relevant problem.**
- Good luck!

This page is for grading purposes only.

Problem	Points	Score
1-10	20	
11	10	
12	10	
13	10	
14	10	
15	15	
Total	**75**	

Answer Sheet for Multiple Choice Questions

Problem	Answer
1	E
2	B
3	A
4	C
5	D
6	A
7	C
8	B
9	A
10	E

Multiple Choice Questions

For problems 1-10, no justification is required and each problem is worth 2 points.

1. (2 points) The value of

$$\int_e^{e^4} \frac{dx}{x\sqrt{\ln(x)}}$$

is

   A. -2

   B. -1

   C. 0

   D. 1

   E. 2

2. (2 points) What is the area under the curve $y = \frac{1}{x}$ between $x = 1$ and $x = 6$?

   A. $1 - \frac{1}{36}$

   B. $\ln(6)$

   C. $-\ln(6)$

   D. 0

   E. None of the above.

3. (2 points) What is the value of

$$\int_4^9 \ln(\sqrt(x))\, dx?$$

A. $9\ln(3) - 4\ln(2) - \frac{5}{2}$

B. $9\ln(3) - 4\ln(2) - \frac{13}{2}$

C. $9\ln(3) - 4\ln(2) - 5$

D. $9\ln(3) - 4\ln(2) - 11$

E. None of the above.

4. (2 points) What is the value of

$$\int_0^1 (2x+1)e^{x^2+x}\, dx?$$

A. e^2

B. $e^2 + 1$

C. $e^2 - 1$

D. $1 - e^2$

E. None of the above.

5. (2 points) Which one of these functions has no horizontal asymptote?

A.
$$g(x) = \frac{x-6}{x^2+2}$$

B.
$$h(x) = \frac{x-9}{x+3}$$

C.
$$f(x) = \frac{x^2}{1-3x^2}$$

D.
$$p(x) = \frac{x^3-2}{6x^2-5}$$

E. All of these functions have a horizontal asymptote.

6. (2 points) A honeybee population starts with 100 bees and increases at a rate of $n'(t)$ bees per week. What does

$$100 + \int_0^{15} n'(t)\,dt$$

represent?

   A. The total bee population after 15 weeks.

   B. The increase in the bee population after 15 weeks.

   C. The decrease in the bee population after 15 weeks.

   D. The rate of change of the bee population.

   E. None of the above.

7. (2 points) What kind of discontinuities does the function

$$f(x) = \frac{x^3 + 3x^2 - 4x}{x^2 + x - 12}$$

have?

   A. $f(x)$ has one jump discontinuity and one infinite discontinuity.

   B. $f(x)$ has one jump discontinuity and one removable discontinuity.

   C. $f(x)$ **has one removable discontinuity and one infinite discontinuity.**

   D. $f(x)$ has no discontinuities.

   E. None of the above.

8. (2 points) Assuming that y is a function of x, find $\frac{dy}{dx}$ at the point $(0, 1)$ if $x - y^4 = (x + 2)(y + x^2)$.

   A. -1

   B. 0

   C. 1

   D. 2

   E. None of the above.

9. (2 points) Two cars are driving away from an intersection in perpendicular directions. The first car's velocity is 7 meters per second and the second car's velocity is 3 meters per second. At a certain instant, the first car is 5 meters from the intersection and the second car is 12 meters from the intersection. What is the rate of change of the distance between the cars at that instant (in meters per second)?

   A. 71/13

   B. 99/13

   C. 13

   D. $\sqrt{58}$

   E. None of the above.

10. (2 points) Let $f(x) = -\frac{1}{2}x^4 - 6x^3 - 27x^2$. For what values of x does the graph of f have a point of inflection?

   A. $x = 3$.

   B. Both $x = 3$ and $x = -3$.

   C. $x = \frac{3}{2}$.

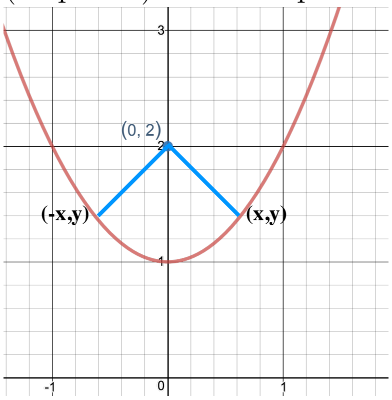
   D. $x = -3$.

   E. None of the above

Long Answer Questions

For each long answer problem, you must give a complete explanation and provide full justification of your answer to receive full credit. Write clearly in complete sentences.

11. (10 points) Find the points on the curve $y = x^2 + 1$ that are closest to the point $(0, 2)$.



Solution: Step 1: Build a mathematical Model.

The distance between a point (x, y) on the curve and $(0, 2)$ is given by the formula:

$$d = \sqrt{(x-0)^2 + (y-2)^2} = \sqrt{x^2 + (y-2)^2}.$$

Our minimization is summarized as

$$\text{MINIMIZE:} \quad d = \sqrt{x^2 + (y-2)^2} \tag{1}$$
$$\text{CONSTRAINT:} \quad y = x^2 + 1. \tag{2}$$

Step 2: Eliminate one of the variables from the minimize/maximize equation in the model using the constraint equation.

Eliminating x from Eq. (1) using Eq. (2)

$y = x^2 + 1 \implies x^2 = y - 1 \implies d = \sqrt{y - 1 + (y-2)^2} = \sqrt{y^2 - 3y + 3}$

Step 3: Choose a method for finding maxima and minima.

We will use the first derivative test.

Step 4: Find the critical numbers/points.

As $d(y) = \sqrt{y^2 - 3y + 3} \implies d'(y) = \frac{1}{2}\frac{2y-3}{\sqrt{y^2-3y+3}}$. We find critical numbers by solving $d'(y) = 0 \implies 2y - 3 = 0 \implies \boxed{y = \dfrac{3}{2}}$

Step 5: Perform a Max/Min test.

As $d'(y) < 0$ for all $y < \dfrac{3}{2}$ and $d'(y) > 0$ for all $y > \dfrac{3}{2}$, the first derivative test applies and we conclude that absolute minimum of d occurs at $y = \dfrac{3}{2}$.

Step 6: State your results

The absolute minimum value of $d(y)$ is $d\left(\dfrac{3}{2}\right) = \dfrac{\sqrt{3}}{2}$.

The corresponding value of x can be found by substituting $y = \dfrac{3}{2}$ in $y = x^2 + 1$ $\implies x = \pm\sqrt{y-1} = \pm\sqrt{\dfrac{3}{2} - 1} \implies \boxed{x = \pm\dfrac{1}{\sqrt{2}}}$.

Thus, the closest points from $(0, 2)$ to the curve $y = x^2 + 1$ are $(-\dfrac{1}{\sqrt{2}}, \dfrac{3}{2})$ and $(+\dfrac{1}{\sqrt{2}}, \dfrac{3}{2})$ and the closest distance is $\dfrac{\sqrt{3}}{2}$.

12. (10 points total) Evaluate the following limits.

(a) (5 points)

$$\lim_{x \to 2} \frac{\sqrt{x} - 2}{x - 4}$$

Solution: To solve this problem, we recognize the function as the quotient of two realtively simple functions, $\sqrt{x} - 2$ and the polynomial $x - 4$. By one of our theorems, the quotient is continuous on its domain, which is all real numbers except $\{x < 0\}$ and $x = 4$. Since, $x = 2$ is within the domain, the quotient is continuous there and we may find the limit by direct substitution:

$$\lim_{x \to 2} \frac{\sqrt{x} - 2}{x - 4} = \frac{\sqrt{2} - 2}{2 - 4} = -\frac{\sqrt{2} - 2}{2}.$$

(b) (5 points)

$$\lim_{y \to 2^-} f(y)$$

where

$$f(y) = \begin{cases} y + 1 & \text{if } y < 2 \\ y^2 - 4 & \text{if } y \geq 2 \end{cases}$$

Solution: As we are taking the limit as x approaches two from below, the relevant part of the definition of the function is that $f(y) = y + 1$ when $y < 2$. Since, $y + 1$ is a polynomial, it is continuous so we have $\lim_{x \to 2^-} y + 1 = 3$ by direct substitution.

13. (10 points) Use a left-endpoint Riemann sum with six rectangles to estimate the value of $\int_0^2 (3x+2)\, dx$.

Solution: We have $n = 6, a = 0, b = 2, \Delta x = \frac{b-a}{n} = \frac{1}{3}$, and the values y_i correspond to the height of the graph of $y = 3x + 2$ at the left edge of each interval. The left end points are $0, 1/3, 2/3, 1, 4/3, 1/3$ and $5/3$ and the y values are $2, 3, 4, 5, 6$ and 7. Therefore, the left Riemann tells that the area under the graph of $3x + 2$ between $a = 0$ and $b = 2$ is approximately:

$$(y_0 + y_1 + \cdot + y_{n-1})\Delta x = (2 + 3 + 4 + 5 + 6 + 7).\frac{1}{3} = 9.$$

Below $\int_0^2 (3x+2)dx$ is the area of a trapezoid with width 2 and sides of height 2 and 8, we can easily check our work:

$$\int_0^2 (3x+2)dx = 2.\frac{2+8}{2} = 10.$$

From the figure we see that the left Riemann sum slightly underestimates the area, so our answer of 9 is probably correct.

14. (10 points) Using the formal definition of the derivative find $f'(1)$ where

$$f(x) = x^2 - 3x + 10.$$

Solution: The definition of the derivative is

$$f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}.$$

With this function, $f(x) - f(1) = 2x^2 - 3x + 10 - (2 - 4 + 10) = x^2 - 3x + 2$. Since $x^2 - 3x + 2 = (x - 2)(x - 1)$, we have

$$f'(1) = \lim_{x \to 1} \frac{(x - 2)(x - 1)}{x - 1}.$$

When we evaluate a limit, we evaluate the function nearbut not equal to the desired point which, in this case, is $x = 1$. Consequently, we can cancel the factors of $x - 1$. So,

$$f'(1) = \lim_{x \to 1} \frac{(x - 2)(x - 1)}{x - 1} = \lim_{x \to 1} x - 2.$$

Since $x - 2$ is a polynomial, and hence continuous, we can evaluate the limit by direct substitution:

$$f'(1) = \lim_{x \to 1} x - 2 = 1 - 2 = -1.$$

15. (15 points total) Evaluate the following expressions:

(a) (5 points)

$$\int_0^{\pi} x\sin(x)\cos(x)\; dx$$

Hint: $\sin^2(x) = \frac{1-\cos(2x)}{2}$.

Solution: For this problem, we use integration by parts with $u = x$ and $dv = \sin(x)\cos(x)dx$. Then, by the power rule, $du = dx$. To integrate dv, we make the substitution $z = \sin(x)$. Then, $dx = \cos(x)dx$ and

$$\int \sin(x)\cos(x)dx = \int zdz = \frac{z^2}{2} + C = \frac{\sin^2(x)}{2} + C$$

So, we can take $v = \frac{\sin^2(x)}{2}$ and the integration by parts formula yields

$$\begin{aligned}\int_0^{\pi} x\sin(x)\cos(x)\; dx &= \left(x\frac{\sin^2(x)}{2}\right)\Bigg|_0^{\pi} - \int_0^{\pi} \frac{\sin^2(x)}{2}\; dx \\ &= \pi\frac{\sin^2(\pi)}{2} - 0\frac{\sin^2(0)}{2} - \int_0^{\pi} \frac{\sin^2(x)}{2}\; dx \\ &= -\int_0^{\pi} \frac{\sin^2(x)}{2}\; dx\end{aligned}$$

Using the hint, we have

$$\begin{aligned}\int_0^{\pi} \frac{\sin^2(x)}{2}\; dx &= \frac{1}{2}\int_0^{\pi} \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)\; dx \\ &= \left(\frac{1}{4}x\right)\Bigg|_0^{\pi} - \frac{1}{2}\int_0^{\pi} \cos(2x)\; dx \\ &= \frac{\pi}{4} - 0 - \int_0^{\pi} \cos(2x)\; dx\end{aligned}$$

For the last integral, we use a substitution $z = 2x$ which makes $dz = 2dx$. Then,

$$\int \cos(2x)\; dx = \frac{1}{2}\int \cos(z)dz = \frac{1}{2}\sin(z) + C = \frac{1}{2}\cos(2x) + C.$$

Using this anti-derivative,

$$\int_0^{\pi} \cos(2x)\; dx = \frac{1}{2}\cos(2\pi) - \frac{1}{2}\cos(0) = \frac{1}{2} - \frac{1}{2} = 0.$$

Putting all this together yields,

$$\int \sin(x)\cos(x)dx = -\frac{\pi}{4}.$$

(b) (5 points)

$$\frac{d}{dx}\int_{-12}^{x^2+2x+1} \sin(\ln(t^2+1))\, dt$$

Solution: We can use the first part of the Fundamental Theorem of calculus, in conjunction with the chain rule to solve this problem. If we write

$$\int_{-12}^{x^2+2x+1} \sin(\ln(t^2+1))\, dt = f(g(x)),$$

where $f(x) = \int_{-12}^{x} \sin(\ln(t^2+1))\, dt$ and $g(x) = x^2+2x+1$ then

$$\frac{d}{dx} \sin(\ln(t^2+1))\, dt = f'(g(x))g'(x).$$

By the Fundamental Theorem of Calculus, $f'(x) = \sin(\ln(x^2+1))$ and by the power rule, $g'(x) = 2x+2$. So,

$$\frac{d}{dx} \sin(\ln(t^2+1))\, dt = f'(g(x))g'(x) = \sin(\ln((x^2+2x+1)^2+1))(2x+2).$$

(c) (5 points)

$$\int_0^{\frac{\pi}{4}} \frac{\sin(\theta)}{\cos^4(\theta)}\, d\theta$$

Solution: We can solve this problem using substitution with $u = \cos(\theta)$. Then, $du = -\sin(\theta)d\theta$ and

$$\int \frac{\sin(\theta)}{\cos^4(\theta)}\, d\theta = -\int u^{-4}\, du = \frac{1}{3}u^{-3} + C = \frac{1}{3}(\cos(\theta))^{-3} + C.$$

Using this anti-derivative and the second part of the Fundamental Theorem of Calculus, we get

$$\int_0^{\frac{\pi}{4}} \frac{\sin(\theta)}{\cos^4(\theta)}\, d\theta = \frac{1}{3}(\cos(\pi/4))^{-3} - \frac{1}{3}(\cos(0))^{-3} = \frac{1}{3}\left(\frac{\sqrt{2}}{2}\right)^{-3} - \frac{1}{3}.$$

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