Math 3 Winter 2018

Midterm 1

January 25, 2018

Your name (please print): ____

Circle your section:

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Instructions:

- This exam is closed book and closed notes. **No calculators, phones, or other devices may be used during the exam.**
- **The Honor Principle requires that you may not give or receive any help during the exam, though you may ask instructors for clarification.**
- This exam consists of **14** questions and you have **two (2)** hours to complete them. Point values are indicated for each question.
  - For multiple choice questions, no justification is required.
  - For long answer questions, you must justify all of your answers completely and show your work to receive full credit. Write clearly in complete sentences. Partial credit may be assigned to incomplete solutions.
- Please do all your work in this exam booklet. There are five (5) pages for extra work at the end of the booklet. **No work on the extra pages will be considered unless you specifically reference it on the relevant problem.**
- Good luck!

This page is for grading purposes only.

Problem	Points	Score
1-10	20	
11	10	
12	10	
13	10	
14	10	
Total	**60**	

Answer Sheet for Multiple Choice Questions

Problem	Answer
1	B
2	A
3	D
4	C
5	B
6	A
7	C
8	E
9	D
10	A

Multiple Choice Questions

For problems 1-10, no justification is required and each problem is worth 2 points.

1. What is the average rate of change of the function $g(x) = \sqrt{x}$ over the interval $5 \leq x \leq t$?

   (a)
   $$\frac{\sqrt{t-5}}{t}$$

   (b)
   $$\frac{\sqrt{t}-\sqrt{5}}{t-5}$$

   (c)
   $$\frac{\sqrt{t-5}}{t-5}$$

   (d)
   $$\frac{\sqrt{t}-\sqrt{5}}{5}$$

   (e) None of the above.

2. What is
   $$\lim_{x\to 1} \frac{\sqrt{5x+4}-3}{x-1}?$$

   (a) 5/6

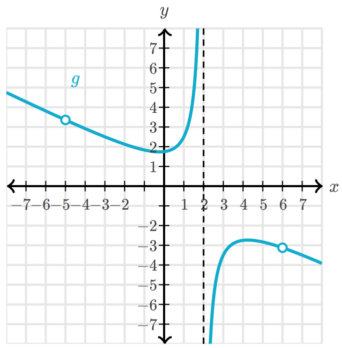
   (b) 3/5

   (c) 1

   (d) The limit does not exist.

   (e) None of the above.

3. For the function in the graph shown below, which of the following statements are true?



(a) The function is continuous for all values of x shown in the graph.

(b) There is a single discontinuity at $x = 2$ and it is an infinite discontinuity.

(c) The function has at least one jump discontinuity.

(d) The function has two removable discontinuities at $x = -5$ and at $x = 6$.

(e) None of the statements above are true.

4. Which derivative is described by the following expression?

$$\lim_{x \to 0} \frac{\cos(x) - 1}{x - 0}$$

(a) $g'(0)$, where $g(x) = \frac{\cos(x)-1}{x}$.

(b) $g'(1)$, where $g(x) = \cos(x)$.

(c) $g'(0)$, where $g(x) = \cos(x)$.

(d) $g'(1)$, where $g(x) = \frac{\cos(x)-1}{x}$.

5. Suppose $f(x)$ is a continuous function on $x \in [-1, 3]$, and $f(-1) = 4$, $f(3) = 7$. By the Intermediate Value Theorem, we can conclude that

(a) there exist a number $c \in (-1, 3)$ such that $f(c) = 0$

(b) there exist a number $c \in (-1, 3)$ such that $f(c) = 5$

(c) there exist a number $c \in (4, 7)$ such that $f(c) = 0$

(d) there exist a number $c \in (4, 7)$ such that $f(c) = 5$

(e) there exist a number $c \in (-1, 7)$ such that $f(c) = 0$

6. For $f(x) = x^2 - 3x$, $f'(1)$ is equal to

   (a) -1

   (b) 0

   (c) 1

   (d) 3

   (e) None of the above.

7. Find
$$\lim_{x \to -3} \frac{4x^2 + 12x}{x^2 + 4x + 3}.$$

   (a) -3

   (b) -12

   (c) 6

   (d) The limit does not exist.

   (e) None of the above.

8. A definition of the derivative of $f(x) = \sqrt{2x - 5}$ is

   (a)
$$f'(x) = \lim_{h \to a} \frac{\sqrt{2x + h - 5} - \sqrt{2x - 5}}{h}$$

   (b)
$$f'(x) = \lim_{h \to 0} \frac{\sqrt{2x + 2h - 5} + \sqrt{2x - 5}}{h}$$

   (c)
$$f'(x) = \lim_{h \to 0} \frac{\sqrt{2x + h - 5} - \sqrt{2x - 5}}{h}$$

   (d)
$$f'(x) = \lim_{h \to a} \frac{\sqrt{2x + h - 5} + \sqrt{2x - 5}}{h}$$

   (e)
$$f'(x) = \lim_{h \to 0} \frac{\sqrt{2x + 2h - 5} - \sqrt{2x - 5}}{h}$$

9. If $a_n = \dfrac{n^2+6n-2}{2n^2+3n-1}$ defines a sequence for $n = 1, 2, 3, \ldots$, then the limit of the sequence as n tend to infinity is

   (a) 2

   (b) 1

   (c) ∞

   (d) 1/2

   (e) None of the above.

10. If $\lim_{x\to -1} f(x) = 1$ and $\lim_{x\to 1} g(x) = -2$ then $\lim_{x\to -1} g(f(x)) + f(x)^2$ equals

   (a) -1

   (b) 1

   (c) 0

   (d) The limit does not exist.

   (e) None of the above.

Long Answer Questions

For each long answer problem, you must give a complete explanation and provide full justification of your answer to receive full credit. Write clearly in complete sentences.

11. (10 points total)

    (a) (5 points) State the definition of covergence of a sequence.

    Solution: A sequence (a_n) converges to c if for every $\epsilon > 0$ there is an index N so that for all $k \geq N$, $|a_k - c| < \epsilon$.

    (b) (5 points) Using the definition from part (a), show that the sequence given by $a_n = 1 - \frac{2}{n}$ for $n = 1, 2, 3, \ldots$ converges to 1.

    Solution: For a fixed $\epsilon > 0$, we begin by solving the inequality in the definition for k to help us pick an N. For this sequence and $c = 1$, this inequality reads $|1 - 2/k - 1| < \epsilon$ or $|-2/k| < \epsilon$. Since k is a positive integer, $|-2/k| = 2/k$ so our original inequality is equivalent to $2/k < \epsilon$. Mulitplying by k and dividing by ϵ yields $2/\epsilon < k$. Picking N to be an integer larger than $2/\epsilon$ therefore guarentees that for any $k \geq N$, $k > 2/\epsilon$ which is equivalent to the desired inequality, $|1-2/k-1| < \epsilon$. Consequently, we have satisfied the definition and conclude the limit equals 1.

12. (10 points) For what value of a is $f(x)$ continuous at x?

$$f(x) = \begin{cases} ax^2 + 2x & x < 2 \\ x^3 - ax & x \geq 2 \end{cases}$$

Solution: We first recall the theorem that states that polynomials are continuous functions. Since for values of x greater than and less than two, the function is defined by polynomial equations, $f(x)$ is continuous at every point except for possibly at $x = 2$. To determine the value of c that makes f continuous at $x = 2$, we evaluate the function and the one-sided limits. If f is continuous there, all these values must be equal. First, by the defintion, we have that $f(2) = 2^3 - a \cdot 2 = 8 - 2a$. Since we evaluate the limit coming from below two using values of x approaching but less than two, we have

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} ax^2 + 2x = 4a + 4.$$

The last equality is true because the polynomial $ax^2 + 2x$ is continuous, so we may compute the limit by evaluating the function at two. Similarly,

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x^3 - ax = 8 - 2a.$$

For all three of these values to be equal, we need $4a + 4 = 8 - 2a$ or $a = 4/6 = 2/3$. With this value of a, $f(x)$ is continuous for all x.

13. (10 points total)

(a) (5 points) State the Squeeze Theorem.

Solution: If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = L$ and for all x near but not equal to a, $f(x) \leq h(x) \leq g(x)$ then $\lim_{x\to a} h(x) = L$.

(b) (5 points) Find $\lim_{x\to 0} \left(x^8 \cos(\frac{1}{x})\right)$.

Solution: We will use the squeeze theorem by first noting that $-1 \leq \cos(1/x) \leq 1$ for all x. Multiplying by x^8 shows that $-x^8 \leq x^8 \cos(1/x) \leq x^8$. Since $\pm x^8$ are polynomials, we can compute their limits by direct subsitution so $\lim_{x\to 0} x^8 = \lim_{x\to 0} -x^8 = 0$. The Squeeze theorem applies and we conclude $\lim_{x\to 0} x^8 \cos(1/x) = 0$.

14. (10 points total)

(a) (5 points) State the definition of continuity of a function, $f(x)$, at the point $x = a$.
Solution: A function $f(x)$ is continuous at $x = a$ if $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = f(a)$.

(b) (5 points) Find and classify all the discontinuities of the function

$$f(x) = \begin{cases} 2 - x^2 & \text{if } x \leq 0 \\ 2 - 2x & \text{if } 0 < x \leq 2 \\ (x-2)^2 - 1 & \text{if } x > 2 \end{cases}$$

Solution: The only values where $f(x)$ might be discontinuous are at $x = 0$ and at $x = 2$, since everywhere else, $f(x)$ is defined by a polynomial, which are continuous.
For $x = 0$, we have:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 2 - x^2 = 2 = f(0) \quad \text{and} \quad \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 2 - 2x = 2.$$

Hence $\lim_{x \to 0} f(x) = 2 = f(0)$ and $f(x)$ is continuous at $x = 0$.
For $x = 2$, we have

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} 2 - 2x = -2 = f(2) \quad \text{and} \quad \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x-2)^2 - 1 = -1.$$

Since the limits from the two sides are different, $f(x)$ has a jump discontinuity at $x = 2$.

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