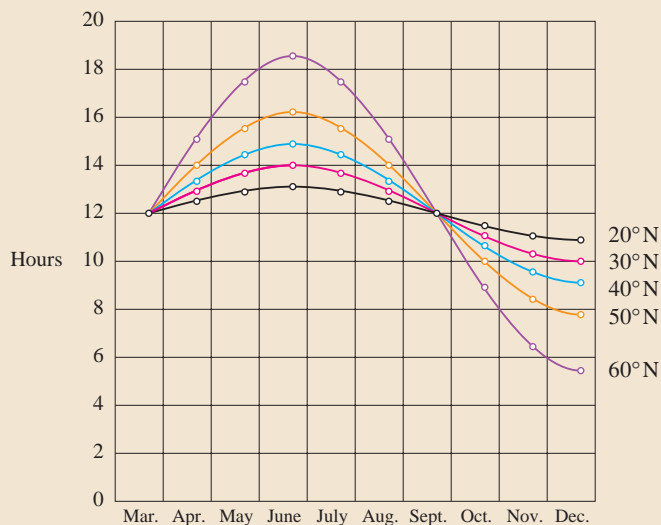


I

FUNCTIONS AND MODELS

A graphical representation of a function—here the number of hours of daylight as a function of the time of year at various latitudes—is often the most natural and convenient way to represent the function.



The fundamental objects that we deal with in calculus are functions. This chapter prepares the way for calculus by discussing the basic ideas concerning functions, their graphs, and ways of transforming and combining them. We stress that a function can be represented in different ways: by an equation, in a table, by a graph, or in words. We look at the main types of functions that occur in calculus and describe the process of using these functions as mathematical models of real-world phenomena. We also discuss the use of graphing calculators and graphing software for computers.

I.1 FOUR WAYS TO REPRESENT A FUNCTION

Functions arise whenever one quantity depends on another. Consider the following four situations.

- A. The area A of a circle depends on the radius r of the circle. The rule that connects r and A is given by the equation $A = \pi r^2$. With each positive number r there is associated one value of A , and we say that A is a *function* of r .
- B. The human population of the world P depends on the time t . The table gives estimates of the world population $P(t)$ at time t , for certain years. For instance,

$$P(1950) \approx 2,560,000,000$$

But for each value of the time t there is a corresponding value of P , and we say that P is a function of t .

- C. The cost C of mailing a first-class letter depends on the weight w of the letter. Although there is no simple formula that connects w and C , the post office has a rule for determining C when w is known.
- D. The vertical acceleration a of the ground as measured by a seismograph during an earthquake is a function of the elapsed time t . Figure 1 shows a graph generated by seismic activity during the Northridge earthquake that shook Los Angeles in 1994. For a given value of t , the graph provides a corresponding value of a .

Year	Population (millions)
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2560
1960	3040
1970	3710
1980	4450
1990	5280
2000	6080

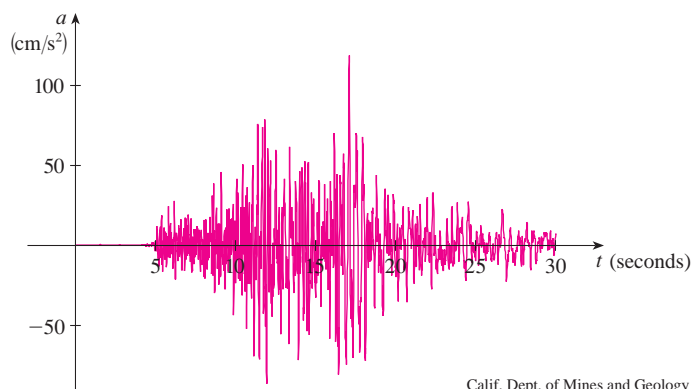


FIGURE 1
Vertical ground acceleration during
the Northridge earthquake

Each of these examples describes a rule whereby, given a number (r , t , w , or t), another number (A , P , C , or a) is assigned. In each case we say that the second number is a function of the first number.

A **function** f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E .

We usually consider functions for which the sets D and E are sets of real numbers. The set D is called the **domain** of the function. The number $f(x)$ is the **value of f at x** and is read “ f of x .” The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain. A symbol that represents an arbitrary number in the *domain* of a function f is called an **independent variable**. A symbol that represents a number in the *range* of f is called a **dependent variable**. In Example A, for instance, r is the independent variable and A is the dependent variable.



FIGURE 2
Machine diagram for a function f

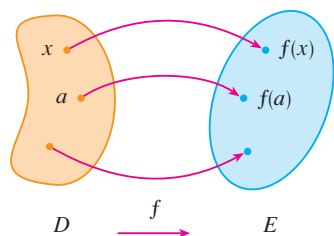


FIGURE 3
Arrow diagram for f

It's helpful to think of a function as a **machine** (see Figure 2). If x is in the domain of the function f , then when x enters the machine, it's accepted as an input and the machine produces an output $f(x)$ according to the rule of the function. Thus we can think of the domain as the set of all possible inputs and the range as the set of all possible outputs.

The preprogrammed functions in a calculator are good examples of a function as a machine. For example, the square root key on your calculator computes such a function. You press the key labeled $\sqrt{}$ (or \sqrt{x}) and enter the input x . If $x < 0$, then x is not in the domain of this function; that is, x is not an acceptable input, and the calculator will indicate an error. If $x \geq 0$, then an *approximation* to \sqrt{x} will appear in the display. Thus the \sqrt{x} key on your calculator is not quite the same as the exact mathematical function f defined by $f(x) = \sqrt{x}$.

Another way to picture a function is by an **arrow diagram** as in Figure 3. Each arrow connects an element of D to an element of E . The arrow indicates that $f(x)$ is associated with x , $f(a)$ is associated with a , and so on.

The most common method for visualizing a function is its graph. If f is a function with domain D , then its **graph** is the set of ordered pairs

$$\{(x, f(x)) \mid x \in D\}$$

(Notice that these are input-output pairs.) In other words, the graph of f consists of all points (x, y) in the coordinate plane such that $y = f(x)$ and x is in the domain of f .

The graph of a function f gives us a useful picture of the behavior or “life history” of a function. Since the y -coordinate of any point (x, y) on the graph is $y = f(x)$, we can read the value of $f(x)$ from the graph as being the height of the graph above the point x (see Figure 4). The graph of f also allows us to picture the domain of f on the x -axis and its range on the y -axis as in Figure 5.

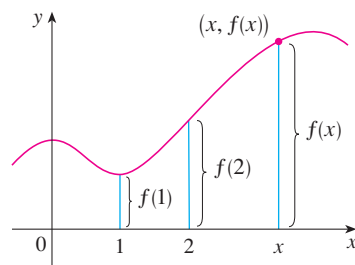


FIGURE 4

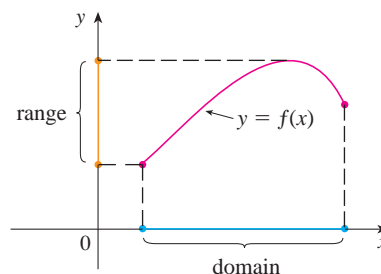


FIGURE 5

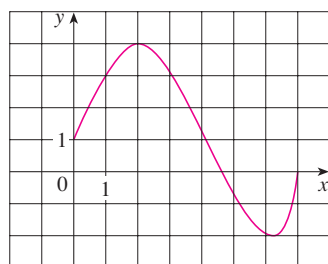


FIGURE 6

EXAMPLE 1 The graph of a function f is shown in Figure 6.

- Find the values of $f(1)$ and $f(5)$.
- What are the domain and range of f ?

SOLUTION

- We see from Figure 6 that the point $(1, 3)$ lies on the graph of f , so the value of f at 1 is $f(1) = 3$. (In other words, the point on the graph that lies above $x = 1$ is 3 units above the x -axis.)

When $x = 5$, the graph lies about 0.7 unit below the x -axis, so we estimate that $f(5) \approx -0.7$.

- We see that $f(x)$ is defined when $0 \leq x \leq 7$, so the domain of f is the closed interval $[0, 7]$. Notice that f takes on all values from -2 to 4 , so the range of f is

$$\{y \mid -2 \leq y \leq 4\} = [-2, 4]$$

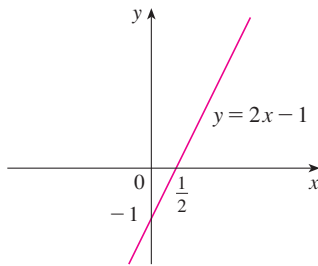


FIGURE 7

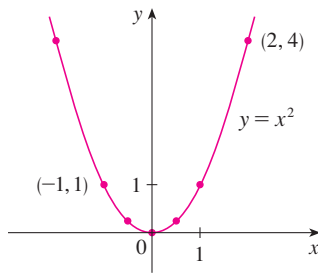


FIGURE 8

EXAMPLE 2 Sketch the graph and find the domain and range of each function.

(a) $f(x) = 2x - 1$

(b) $g(x) = x^2$

SOLUTION

(a) The equation of the graph is $y = 2x - 1$, and we recognize this as being the equation of a line with slope 2 and y -intercept -1 . (Recall the slope-intercept form of the equation of a line: $y = mx + b$. See Appendix B.) This enables us to sketch a portion of the graph of f in Figure 7. The expression $2x - 1$ is defined for all real numbers, so the domain of f is the set of all real numbers, which we denote by \mathbb{R} . The graph shows that the range is also \mathbb{R} .

(b) Since $g(2) = 2^2 = 4$ and $g(-1) = (-1)^2 = 1$, we could plot the points $(2, 4)$ and $(-1, 1)$, together with a few other points on the graph, and join them to produce the graph (Figure 8). The equation of the graph is $y = x^2$, which represents a parabola (see Appendix C). The domain of g is \mathbb{R} . The range of g consists of all values of $g(x)$, that is, all numbers of the form x^2 . But $x^2 \geq 0$ for all numbers x and any positive number y is a square. So the range of g is $\{y \mid y \geq 0\} = [0, \infty)$. This can also be seen from Figure 8. ■

EXAMPLE 3 If $f(x) = 2x^2 - 5x + 1$ and $h \neq 0$, evaluate $\frac{f(a+h) - f(a)}{h}$.

SOLUTION We first evaluate $f(a+h)$ by replacing x by $a+h$ in the expression for $f(x)$:

$$\begin{aligned} f(a+h) &= 2(a+h)^2 - 5(a+h) + 1 \\ &= 2(a^2 + 2ah + h^2) - 5(a+h) + 1 \\ &= 2a^2 + 4ah + 2h^2 - 5a - 5h + 1 \end{aligned}$$

Then we substitute into the given expression and simplify:

$$\begin{aligned} \frac{f(a+h) - f(a)}{h} &= \frac{(2a^2 + 4ah + 2h^2 - 5a - 5h + 1) - (2a^2 - 5a + 1)}{h} \\ &= \frac{2a^2 + 4ah + 2h^2 - 5a - 5h + 1 - 2a^2 + 5a - 1}{h} \\ &= \frac{4ah + 2h^2 - 5h}{h} = 4a + 2h - 5 \end{aligned}$$

■ The expression

$$\frac{f(a+h) - f(a)}{h}$$

in Example 3 is called a **difference quotient** and occurs frequently in calculus. As we will see in Chapter 2, it represents the average rate of change of $f(x)$ between $x = a$ and $x = a + h$.

REPRESENTATIONS OF FUNCTIONS

There are four possible ways to represent a function:

- verbally (by a description in words)
- numerically (by a table of values)
- visually (by a graph)
- algebraically (by an explicit formula)

If a single function can be represented in all four ways, it's often useful to go from one representation to another to gain additional insight into the function. (In Example 2, for instance, we started with algebraic formulas and then obtained the graphs.) But certain

functions are described more naturally by one method than by another. With this in mind, let’s reexamine the four situations that we considered at the beginning of this section.

- A. The most useful representation of the area of a circle as a function of its radius is probably the algebraic formula $A(r) = \pi r^2$, though it is possible to compile a table of values or to sketch a graph (half a parabola). Because a circle has to have a positive radius, the domain is $\{r \mid r > 0\} = (0, \infty)$, and the range is also $(0, \infty)$.
- B. We are given a description of the function in words: $P(t)$ is the human population of the world at time t . The table of values of world population provides a convenient representation of this function. If we plot these values, we get the graph (called a *scatter plot*) in Figure 9. It too is a useful representation; the graph allows us to absorb all the data at once. What about a formula? Of course, it’s impossible to devise an explicit formula that gives the exact human population $P(t)$ at any time t . But it is possible to find an expression for a function that *approximates* $P(t)$. In fact, using methods explained in Section 1.2, we obtain the approximation

$$P(t) \approx f(t) = (0.008079266) \cdot (1.013731)^t$$

and Figure 10 shows that it is a reasonably good “fit.” The function f is called a *mathematical model* for population growth. In other words, it is a function with an explicit formula that approximates the behavior of our given function. We will see, however, that the ideas of calculus can be applied to a table of values; an explicit formula is not necessary.

Year	Population (millions)
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2560
1960	3040
1970	3710
1980	4450
1990	5280
2000	6080

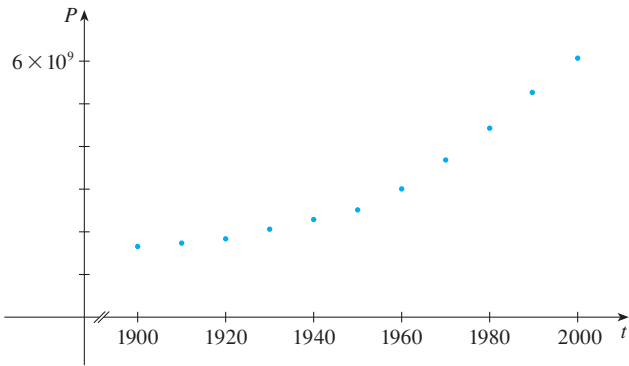


FIGURE 9

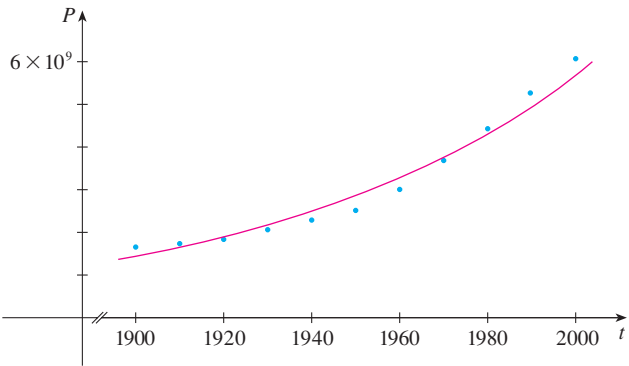


FIGURE 10

■ A function defined by a table of values is called a *tabular* function.

w (ounces)	$C(w)$ (dollars)
$0 < w \leq 1$	0.39
$1 < w \leq 2$	0.63
$2 < w \leq 3$	0.87
$3 < w \leq 4$	1.11
$4 < w \leq 5$	1.35
\vdots	\vdots
$12 < w \leq 13$	3.27

The function P is typical of the functions that arise whenever we attempt to apply calculus to the real world. We start with a verbal description of a function. Then we may be able to construct a table of values of the function, perhaps from instrument readings in a scientific experiment. Even though we don’t have complete knowledge of the values of the function, we will see throughout the book that it is still possible to perform the operations of calculus on such a function.

- C. Again the function is described in words: $C(w)$ is the cost of mailing a first-class letter with weight w . The rule that the US Postal Service used as of 2007 is as follows: The cost is 39 cents for up to one ounce, plus 24 cents for each successive ounce up to 13 ounces. The table of values shown in the margin is the most convenient representation for this function, though it is possible to sketch a graph (see Example 10).
- D. The graph shown in Figure 1 is the most natural representation of the vertical acceleration function $a(t)$. It’s true that a table of values could be compiled, and it is even

possible to devise an approximate formula. But everything a geologist needs to know—amplitudes and patterns—can be seen easily from the graph. (The same is true for the patterns seen in electrocardiograms of heart patients and polygraphs for lie-detection.)

In the next example we sketch the graph of a function that is defined verbally.

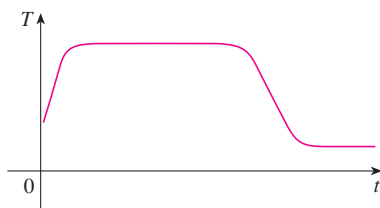


FIGURE 11

EXAMPLE 4 When you turn on a hot-water faucet, the temperature T of the water depends on how long the water has been running. Draw a rough graph of T as a function of the time t that has elapsed since the faucet was turned on.

SOLUTION The initial temperature of the running water is close to room temperature because the water has been sitting in the pipes. When the water from the hot-water tank starts flowing from the faucet, T increases quickly. In the next phase, T is constant at the temperature of the heated water in the tank. When the tank is drained, T decreases to the temperature of the water supply. This enables us to make the rough sketch of T as a function of t in Figure 11.

In the following example we start with a verbal description of a function in a physical situation and obtain an explicit algebraic formula. The ability to do this is a useful skill in solving calculus problems that ask for the maximum or minimum values of quantities.

EXAMPLE 5 A rectangular storage container with an open top has a volume of 10 m^3 . The length of its base is twice its width. Material for the base costs \$10 per square meter; material for the sides costs \$6 per square meter. Express the cost of materials as a function of the width of the base.

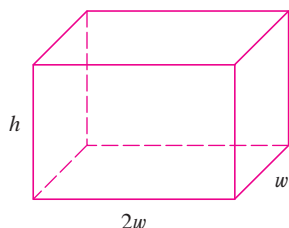


FIGURE 12

SOLUTION We draw a diagram as in Figure 12 and introduce notation by letting w and $2w$ be the width and length of the base, respectively, and h be the height.

The area of the base is $(2w)w = 2w^2$, so the cost, in dollars, of the material for the base is $10(2w^2)$. Two of the sides have area wh and the other two have area $2wh$, so the cost of the material for the sides is $6[2(wh) + 2(2wh)]$. The total cost is therefore

$$C = 10(2w^2) + 6[2(wh) + 2(2wh)] = 20w^2 + 36wh$$

To express C as a function of w alone, we need to eliminate h and we do so by using the fact that the volume is 10 m^3 . Thus

$$w(2w)h = 10$$

which gives

$$h = \frac{10}{2w^2} = \frac{5}{w^2}$$

Substituting this into the expression for C , we have

$$C = 20w^2 + 36w\left(\frac{5}{w^2}\right) = 20w^2 + \frac{180}{w}$$

Therefore, the equation

$$C(w) = 20w^2 + \frac{180}{w} \quad w > 0$$

expresses C as a function of w .

■ In setting up applied functions as in Example 5, it may be useful to review the principles of problem solving as discussed on page 76, particularly *Step 1: Understand the Problem*.

EXAMPLE 6 Find the domain of each function.

(a) $f(x) = \sqrt{x + 2}$

(b) $g(x) = \frac{1}{x^2 - x}$

SOLUTION

(a) Because the square root of a negative number is not defined (as a real number), the domain of f consists of all values of x such that $x + 2 \geq 0$. This is equivalent to $x \geq -2$, so the domain is the interval $[-2, \infty)$.

(b) Since

$$g(x) = \frac{1}{x^2 - x} = \frac{1}{x(x - 1)}$$

and division by 0 is not allowed, we see that $g(x)$ is not defined when $x = 0$ or $x = 1$. Thus the domain of g is

$$\{x \mid x \neq 0, x \neq 1\}$$

which could also be written in interval notation as

$$(-\infty, 0) \cup (0, 1) \cup (1, \infty)$$

The graph of a function is a curve in the xy -plane. But the question arises: Which curves in the xy -plane are graphs of functions? This is answered by the following test.

THE VERTICAL LINE TEST A curve in the xy -plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.

The reason for the truth of the Vertical Line Test can be seen in Figure 13. If each vertical line $x = a$ intersects a curve only once, at (a, b) , then exactly one functional value is defined by $f(a) = b$. But if a line $x = a$ intersects the curve twice, at (a, b) and (a, c) , then the curve can't represent a function because a function can't assign two different values to a .

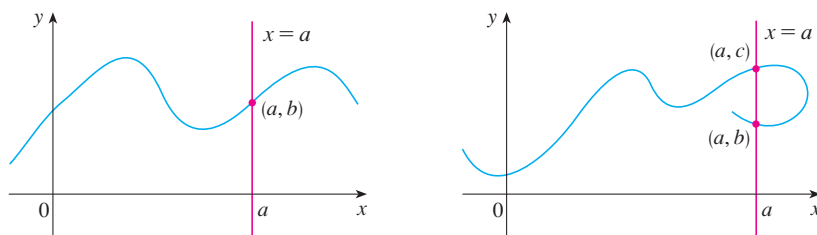


FIGURE 13

For example, the parabola $x = y^2 - 2$ shown in Figure 14(a) on the next page is not the graph of a function of x because, as you can see, there are vertical lines that intersect the parabola twice. The parabola, however, does contain the graphs of *two* functions of x . Notice that the equation $x = y^2 - 2$ implies $y^2 = x + 2$, so $y = \pm\sqrt{x + 2}$. Thus the upper and lower halves of the parabola are the graphs of the functions $f(x) = \sqrt{x + 2}$ [from Example 6(a)] and $g(x) = -\sqrt{x + 2}$. [See Figures 14(b) and (c).] We observe that if we reverse the roles of x and y , then the equation $x = h(y) = y^2 - 2$ does define x as a function of y (with y as the independent variable and x as the dependent variable) and the parabola now appears as the graph of the function h .

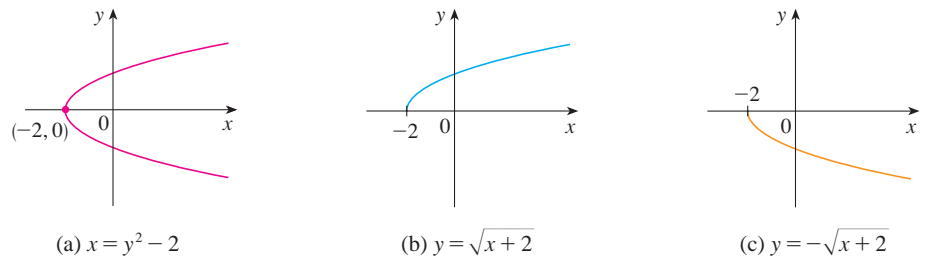


FIGURE 14

PIECEWISE DEFINED FUNCTIONS

The functions in the following four examples are defined by different formulas in different parts of their domains.

EXAMPLE 7 A function f is defined by

$$f(x) = \begin{cases} 1 - x & \text{if } x \leq 1 \\ x^2 & \text{if } x > 1 \end{cases}$$

Evaluate $f(0)$, $f(1)$, and $f(2)$ and sketch the graph.

SOLUTION Remember that a function is a rule. For this particular function the rule is the following: First look at the value of the input x . If it happens that $x \leq 1$, then the value of $f(x)$ is $1 - x$. On the other hand, if $x > 1$, then the value of $f(x)$ is x^2 .

Since $0 \leq 1$, we have $f(0) = 1 - 0 = 1$.

Since $1 \leq 1$, we have $f(1) = 1 - 1 = 0$.

Since $2 > 1$, we have $f(2) = 2^2 = 4$.

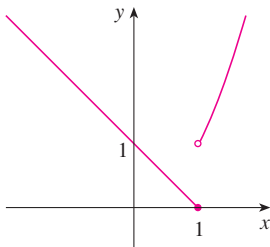


FIGURE 15

How do we draw the graph of f ? We observe that if $x \leq 1$, then $f(x) = 1 - x$, so the part of the graph of f that lies to the left of the vertical line $x = 1$ must coincide with the line $y = 1 - x$, which has slope -1 and y -intercept 1 . If $x > 1$, then $f(x) = x^2$, so the part of the graph of f that lies to the right of the line $x = 1$ must coincide with the graph of $y = x^2$, which is a parabola. This enables us to sketch the graph in Figure 15. The solid dot indicates that the point $(1, 0)$ is included on the graph; the open dot indicates that the point $(1, 1)$ is excluded from the graph. ■

The next example of a piecewise defined function is the absolute value function. Recall that the **absolute value** of a number a , denoted by $|a|$, is the distance from a to 0 on the real number line. Distances are always positive or 0 , so we have

$$|a| \geq 0 \quad \text{for every number } a$$

For example,

$$|3| = 3 \quad |-3| = 3 \quad |0| = 0 \quad |\sqrt{2} - 1| = \sqrt{2} - 1 \quad |3 - \pi| = \pi - 3$$

In general, we have

$$\begin{aligned} |a| &= a & \text{if } a \geq 0 \\ |a| &= -a & \text{if } a < 0 \end{aligned}$$

(Remember that if a is negative, then $-a$ is positive.)

■ For a more extensive review of absolute values, see Appendix A.

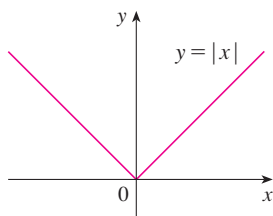


FIGURE 16

EXAMPLE 8 Sketch the graph of the absolute value function $f(x) = |x|$.

SOLUTION From the preceding discussion we know that

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Using the same method as in Example 7, we see that the graph of f coincides with the line $y = x$ to the right of the y -axis and coincides with the line $y = -x$ to the left of the y -axis (see Figure 16). ■

EXAMPLE 9 Find a formula for the function f graphed in Figure 17.

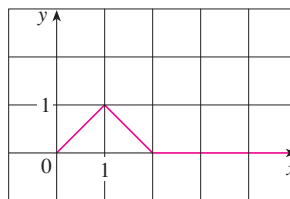


FIGURE 17

SOLUTION The line through $(0, 0)$ and $(1, 1)$ has slope $m = 1$ and y -intercept $b = 0$, so its equation is $y = x$. Thus, for the part of the graph of f that joins $(0, 0)$ to $(1, 1)$, we have

$$f(x) = x \quad \text{if } 0 \leq x \leq 1$$

The line through $(1, 1)$ and $(2, 0)$ has slope $m = -1$, so its point-slope form is

$$y - 0 = (-1)(x - 2) \quad \text{or} \quad y = 2 - x$$

So we have

$$f(x) = 2 - x \quad \text{if } 1 < x \leq 2$$

We also see that the graph of f coincides with the x -axis for $x > 2$. Putting this information together, we have the following three-piece formula for f :

$$f(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 2 - x & \text{if } 1 < x \leq 2 \\ 0 & \text{if } x > 2 \end{cases}$$

EXAMPLE 10 In Example C at the beginning of this section we considered the cost $C(w)$ of mailing a first-class letter with weight w . In effect, this is a piecewise defined function because, from the table of values, we have

$$C(w) = \begin{cases} 0.39 & \text{if } 0 < w \leq 1 \\ 0.63 & \text{if } 1 < w \leq 2 \\ 0.87 & \text{if } 2 < w \leq 3 \\ 1.11 & \text{if } 3 < w \leq 4 \\ \vdots & \end{cases}$$

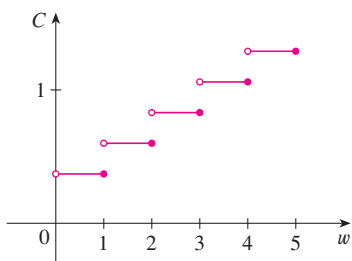


FIGURE 18

The graph is shown in Figure 18. You can see why functions similar to this one are called **step functions**—they jump from one value to the next. Such functions will be studied in Chapter 2. ■

■ Point-slope form of the equation of a line:

$$y - y_1 = m(x - x_1)$$

See Appendix B.

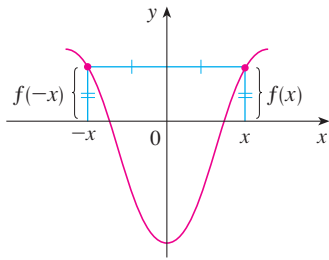


FIGURE 19
An even function

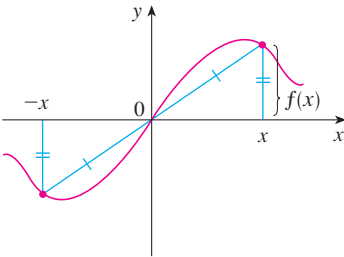


FIGURE 20
An odd function

SYMMETRY

If a function f satisfies $f(-x) = f(x)$ for every number x in its domain, then f is called an **even function**. For instance, the function $f(x) = x^2$ is even because

$$f(-x) = (-x)^2 = x^2 = f(x)$$

The geometric significance of an even function is that its graph is symmetric with respect to the y -axis (see Figure 19). This means that if we have plotted the graph of f for $x \geq 0$, we obtain the entire graph simply by reflecting this portion about the y -axis.

If f satisfies $f(-x) = -f(x)$ for every number x in its domain, then f is called an **odd function**. For example, the function $f(x) = x^3$ is odd because

$$f(-x) = (-x)^3 = -x^3 = -f(x)$$

The graph of an odd function is symmetric about the origin (see Figure 20). If we already have the graph of f for $x \geq 0$, we can obtain the entire graph by rotating this portion through 180° about the origin.

EXAMPLE 11 Determine whether each of the following functions is even, odd, or neither even nor odd.

- (a) $f(x) = x^5 + x$ (b) $g(x) = 1 - x^4$ (c) $h(x) = 2x - x^2$

SOLUTION

$$\begin{aligned} \text{(a)} \quad f(-x) &= (-x)^5 + (-x) = (-1)^5 x^5 + (-x) \\ &= -x^5 - x = -(x^5 + x) \\ &= -f(x) \end{aligned}$$

Therefore f is an odd function.

$$\text{(b)} \quad g(-x) = 1 - (-x)^4 = 1 - x^4 = g(x)$$

So g is even.

$$\text{(c)} \quad h(-x) = 2(-x) - (-x)^2 = -2x - x^2$$

Since $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$, we conclude that h is neither even nor odd. ■

The graphs of the functions in Example 11 are shown in Figure 21. Notice that the graph of h is symmetric neither about the y -axis nor about the origin.

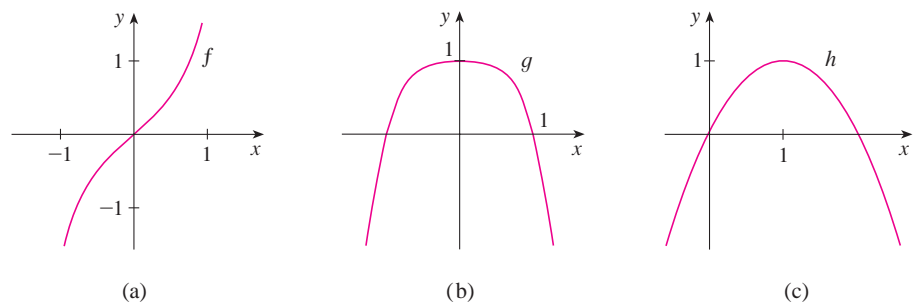


FIGURE 21

INCREASING AND DECREASING FUNCTIONS

The graph shown in Figure 22 rises from A to B , falls from B to C , and rises again from C to D . The function f is said to be increasing on the interval $[a, b]$, decreasing on $[b, c]$, and increasing again on $[c, d]$. Notice that if x_1 and x_2 are any two numbers between a and b with $x_1 < x_2$, then $f(x_1) < f(x_2)$. We use this as the defining property of an increasing function.

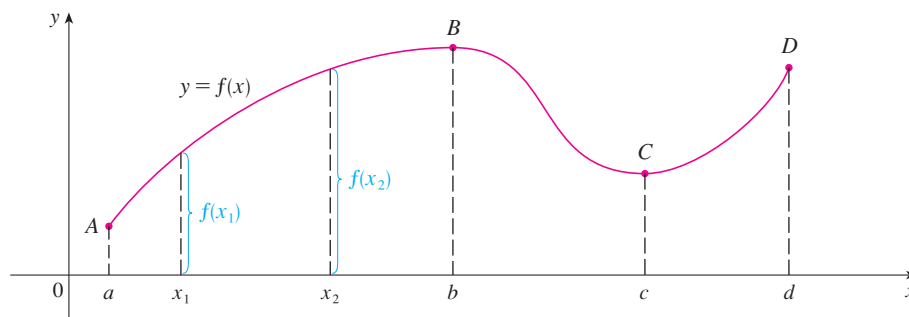


FIGURE 22

A function f is called **increasing** on an interval I if

$$f(x_1) < f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I$$

It is called **decreasing** on I if

$$f(x_1) > f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I$$

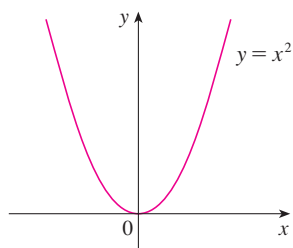


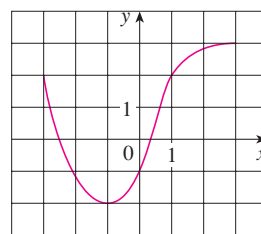
FIGURE 23

In the definition of an increasing function it is important to realize that the inequality $f(x_1) < f(x_2)$ must be satisfied for *every* pair of numbers x_1 and x_2 in I with $x_1 < x_2$.

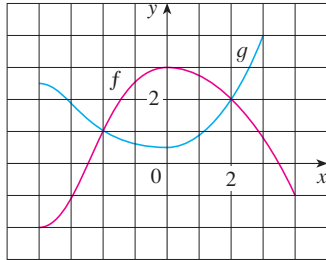
You can see from Figure 23 that the function $f(x) = x^2$ is decreasing on the interval $(-\infty, 0]$ and increasing on the interval $[0, \infty)$.

I.1 EXERCISES

- I. The graph of a function f is given.
 - (a) State the value of $f(-1)$.
 - (b) Estimate the value of $f(2)$.
 - (c) For what values of x is $f(x) = 2$?
 - (d) Estimate the values of x such that $f(x) = 0$.
 - (e) State the domain and range of f .
 - (f) On what interval is f increasing?

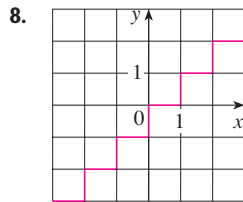
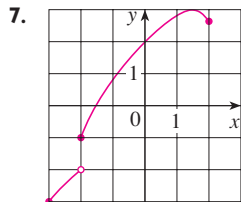
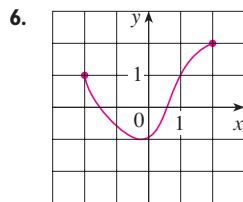
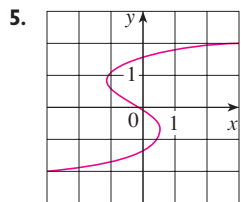


2. The graphs of f and g are given.
- State the values of $f(-4)$ and $g(3)$.
 - For what values of x is $f(x) = g(x)$?
 - Estimate the solution of the equation $f(x) = -1$.
 - On what interval is f decreasing?
 - State the domain and range of f .
 - State the domain and range of g .



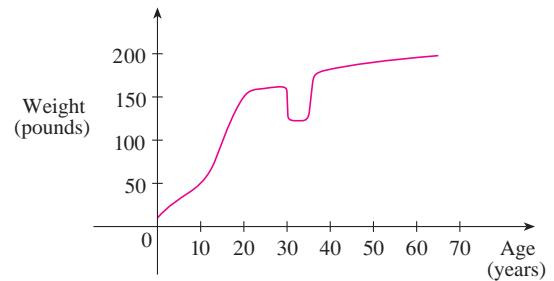
3. Figure 1 was recorded by an instrument operated by the California Department of Mines and Geology at the University Hospital of the University of Southern California in Los Angeles. Use it to estimate the range of the vertical ground acceleration function at USC during the Northridge earthquake.
4. In this section we discussed examples of ordinary, everyday functions: Population is a function of time, postage cost is a function of weight, water temperature is a function of time. Give three other examples of functions from everyday life that are described verbally. What can you say about the domain and range of each of your functions? If possible, sketch a rough graph of each function.

5–8 Determine whether the curve is the graph of a function of x . If it is, state the domain and range of the function.

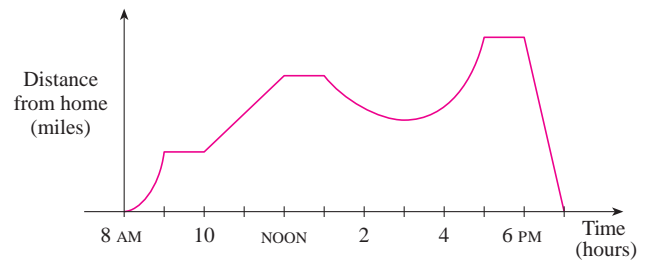


9. The graph shown gives the weight of a certain person as a function of age. Describe in words how this person's weight

varies over time. What do you think happened when this person was 30 years old?



10. The graph shown gives a salesman's distance from his home as a function of time on a certain day. Describe in words what the graph indicates about his travels on this day.



11. You put some ice cubes in a glass, fill the glass with cold water, and then let the glass sit on a table. Describe how the temperature of the water changes as time passes. Then sketch a rough graph of the temperature of the water as a function of the elapsed time.
12. Sketch a rough graph of the number of hours of daylight as a function of the time of year.
13. Sketch a rough graph of the outdoor temperature as a function of time during a typical spring day.
14. Sketch a rough graph of the market value of a new car as a function of time for a period of 20 years. Assume the car is well maintained.
15. Sketch the graph of the amount of a particular brand of coffee sold by a store as a function of the price of the coffee.
16. You place a frozen pie in an oven and bake it for an hour. Then you take it out and let it cool before eating it. Describe how the temperature of the pie changes as time passes. Then sketch a rough graph of the temperature of the pie as a function of time.
17. A homeowner mows the lawn every Wednesday afternoon. Sketch a rough graph of the height of the grass as a function of time over the course of a four-week period.
18. An airplane takes off from an airport and lands an hour later at another airport, 400 miles away. If t represents the time in minutes since the plane has left the terminal building, let $x(t)$ be

the horizontal distance traveled and $y(t)$ be the altitude of the plane.

- Sketch a possible graph of $x(t)$.
- Sketch a possible graph of $y(t)$.
- Sketch a possible graph of the ground speed.
- Sketch a possible graph of the vertical velocity.

19. The number N (in millions) of cellular phone subscribers worldwide is shown in the table. (Midyear estimates are given.)

t	1990	1992	1994	1996	1998	2000
N	11	26	60	160	340	650

- Use the data to sketch a rough graph of N as a function of t .
- Use your graph to estimate the number of cell-phone subscribers at midyear in 1995 and 1999.

20. Temperature readings T (in $^{\circ}\text{F}$) were recorded every two hours from midnight to 2:00 PM in Dallas on June 2, 2001. The time t was measured in hours from midnight.

t	0	2	4	6	8	10	12	14
T	73	73	70	69	72	81	88	91

- Use the readings to sketch a rough graph of T as a function of t .
- Use your graph to estimate the temperature at 11:00 AM.

21. If $f(x) = 3x^2 - x + 2$, find $f(2)$, $f(-2)$, $f(a)$, $f(-a)$, $f(a + 1)$, $2f(a)$, $f(2a)$, $f(a^2)$, $[f(a)]^2$, and $f(a + h)$.

22. A spherical balloon with radius r inches has volume $V(r) = \frac{4}{3}\pi r^3$. Find a function that represents the amount of air required to inflate the balloon from a radius of r inches to a radius of $r + 1$ inches.

- 23–26 Evaluate the difference quotient for the given function. Simplify your answer.

23. $f(x) = 4 + 3x - x^2$, $\frac{f(3 + h) - f(3)}{h}$

24. $f(x) = x^3$, $\frac{f(a + h) - f(a)}{h}$

25. $f(x) = \frac{1}{x}$, $\frac{f(x) - f(a)}{x - a}$

26. $f(x) = \frac{x + 3}{x + 1}$, $\frac{f(x) - f(1)}{x - 1}$

- 27–31 Find the domain of the function.

27. $f(x) = \frac{x}{3x - 1}$

28. $f(x) = \frac{5x + 4}{x^2 + 3x + 2}$

29. $f(t) = \sqrt{t} + \sqrt[3]{t}$

30. $g(u) = \sqrt{u} + \sqrt{4 - u}$

31. $h(x) = \frac{1}{\sqrt[4]{x^2 - 5x}}$

32. Find the domain and range and sketch the graph of the function $h(x) = \sqrt{4 - x^2}$.

- 33–44 Find the domain and sketch the graph of the function.

33. $f(x) = 5$

34. $F(x) = \frac{1}{2}(x + 3)$

35. $f(t) = t^2 - 6t$

36. $H(t) = \frac{4 - t^2}{2 - t}$

37. $g(x) = \sqrt{x - 5}$

38. $F(x) = |2x + 1|$

39. $G(x) = \frac{3x + |x|}{x}$

40. $g(x) = \frac{|x|}{x^2}$

41. $f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ 1 - x & \text{if } x \geq 0 \end{cases}$

42. $f(x) = \begin{cases} 3 - \frac{1}{2}x & \text{if } x \leq 2 \\ 2x - 5 & \text{if } x > 2 \end{cases}$

43. $f(x) = \begin{cases} x + 2 & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}$

44. $f(x) = \begin{cases} x + 9 & \text{if } x < -3 \\ -2x & \text{if } |x| \leq 3 \\ -6 & \text{if } x > 3 \end{cases}$

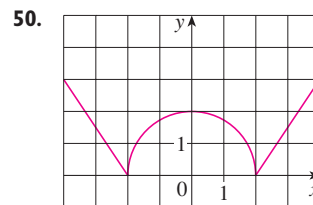
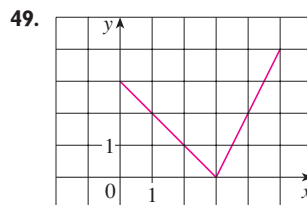
- 45–50 Find an expression for the function whose graph is the given curve.

45. The line segment joining the points $(1, -3)$ and $(5, 7)$

46. The line segment joining the points $(-5, 10)$ and $(7, -10)$

47. The bottom half of the parabola $x + (y - 1)^2 = 0$

48. The top half of the circle $x^2 + (y - 2)^2 = 4$

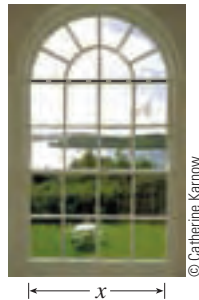


- 51–55 Find a formula for the described function and state its domain.

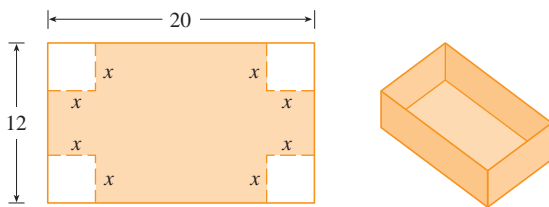
51. A rectangle has perimeter 20 m. Express the area of the rectangle as a function of the length of one of its sides.

52. A rectangle has area 16 m^2 . Express the perimeter of the rectangle as a function of the length of one of its sides.
53. Express the area of an equilateral triangle as a function of the length of a side.
54. Express the surface area of a cube as a function of its volume.
55. An open rectangular box with volume 2 m^3 has a square base. Express the surface area of the box as a function of the length of a side of the base.

56. A Norman window has the shape of a rectangle surmounted by a semicircle. If the perimeter of the window is 30 ft, express the area A of the window as a function of the width x of the window.



57. A box with an open top is to be constructed from a rectangular piece of cardboard with dimensions 12 in. by 20 in. by cutting out equal squares of side x at each corner and then folding up the sides as in the figure. Express the volume V of the box as a function of x .



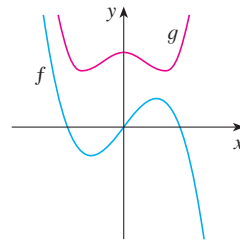
58. A taxi company charges two dollars for the first mile (or part of a mile) and 20 cents for each succeeding tenth of a mile (or part). Express the cost C (in dollars) of a ride as a function of the distance x traveled (in miles) for $0 < x < 2$, and sketch the graph of this function.
59. In a certain country, income tax is assessed as follows. There is no tax on income up to \$10,000. Any income over \$10,000 is taxed at a rate of 10%, up to an income of \$20,000. Any income over \$20,000 is taxed at 15%.
- (a) Sketch the graph of the tax rate R as a function of the income I .

- (b) How much tax is assessed on an income of \$14,000? On \$26,000?
- (c) Sketch the graph of the total assessed tax T as a function of the income I .

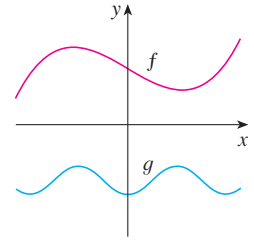
60. The functions in Example 10 and Exercises 58 and 59(a) are called *step functions* because their graphs look like stairs. Give two other examples of step functions that arise in everyday life.

61–62 Graphs of f and g are shown. Decide whether each function is even, odd, or neither. Explain your reasoning.

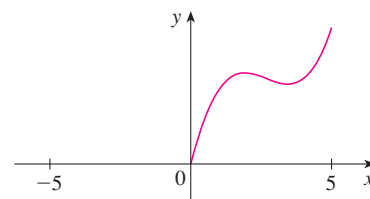
61.



62.



63. (a) If the point $(5, 3)$ is on the graph of an even function, what other point must also be on the graph?
 (b) If the point $(5, 3)$ is on the graph of an odd function, what other point must also be on the graph?
64. A function f has domain $[-5, 5]$ and a portion of its graph is shown.
- (a) Complete the graph of f if it is known that f is even.
 (b) Complete the graph of f if it is known that f is odd.



65–70 Determine whether f is even, odd, or neither. If you have a graphing calculator, use it to check your answer visually.

65. $f(x) = \frac{x}{x^2 + 1}$

66. $f(x) = \frac{x^2}{x^4 + 1}$

67. $f(x) = \frac{x}{x + 1}$

68. $f(x) = x|x|$

69. $f(x) = 1 + 3x^2 - x^4$

70. $f(x) = 1 + 3x^3 - x^5$

1.2 MATHEMATICAL MODELS: A CATALOG OF ESSENTIAL FUNCTIONS

A **mathematical model** is a mathematical description (often by means of a function or an equation) of a real-world phenomenon such as the size of a population, the demand for a product, the speed of a falling object, the concentration of a product in a chemical reaction, the life expectancy of a person at birth, or the cost of emission reductions. The purpose of the model is to understand the phenomenon and perhaps to make predictions about future behavior.

Figure 1 illustrates the process of mathematical modeling. Given a real-world problem, our first task is to formulate a mathematical model by identifying and naming the independent and dependent variables and making assumptions that simplify the phenomenon enough to make it mathematically tractable. We use our knowledge of the physical situation and our mathematical skills to obtain equations that relate the variables. In situations where there is no physical law to guide us, we may need to collect data (either from a library or the Internet or by conducting our own experiments) and examine the data in the form of a table in order to discern patterns. From this numerical representation of a function we may wish to obtain a graphical representation by plotting the data. The graph might even suggest a suitable algebraic formula in some cases.

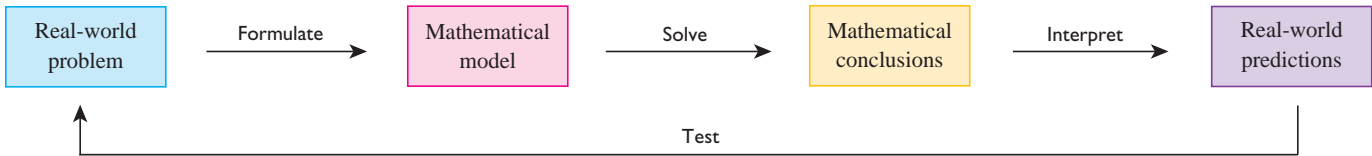


FIGURE 1 The modeling process

The second stage is to apply the mathematics that we know (such as the calculus that will be developed throughout this book) to the mathematical model that we have formulated in order to derive mathematical conclusions. Then, in the third stage, we take those mathematical conclusions and interpret them as information about the original real-world phenomenon by way of offering explanations or making predictions. The final step is to test our predictions by checking against new real data. If the predictions don’t compare well with reality, we need to refine our model or to formulate a new model and start the cycle again.

A mathematical model is never a completely accurate representation of a physical situation—it is an *idealization*. A good model simplifies reality enough to permit mathematical calculations but is accurate enough to provide valuable conclusions. It is important to realize the limitations of the model. In the end, Mother Nature has the final say.

There are many different types of functions that can be used to model relationships observed in the real world. In what follows, we discuss the behavior and graphs of these functions and give examples of situations appropriately modeled by such functions.

LINEAR MODELS

When we say that y is a **linear function** of x , we mean that the graph of the function is a line, so we can use the slope-intercept form of the equation of a line to write a formula for the function as

$$y = f(x) = mx + b$$

where m is the slope of the line and b is the y -intercept.

■ The coordinate geometry of lines is reviewed in Appendix B.

A characteristic feature of linear functions is that they grow at a constant rate. For instance, Figure 2 shows a graph of the linear function $f(x) = 3x - 2$ and a table of sample values. Notice that whenever x increases by 0.1, the value of $f(x)$ increases by 0.3. So $f(x)$ increases three times as fast as x . Thus the slope of the graph $y = 3x - 2$, namely 3, can be interpreted as the rate of change of y with respect to x .

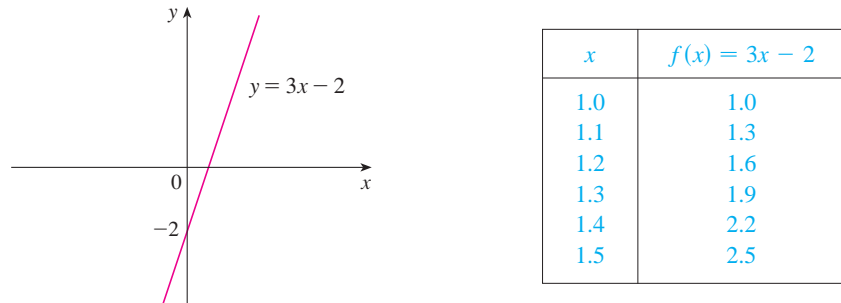


FIGURE 2

EXAMPLE 1

- (a) As dry air moves upward, it expands and cools. If the ground temperature is 20°C and the temperature at a height of 1 km is 10°C , express the temperature T (in $^\circ\text{C}$) as a function of the height h (in kilometers), assuming that a linear model is appropriate.
- (b) Draw the graph of the function in part (a). What does the slope represent?
- (c) What is the temperature at a height of 2.5 km?

SOLUTION

- (a) Because we are assuming that T is a linear function of h , we can write

$$T = mh + b$$

We are given that $T = 20$ when $h = 0$, so

$$20 = m \cdot 0 + b = b$$

In other words, the y-intercept is $b = 20$.

We are also given that $T = 10$ when $h = 1$, so

$$10 = m \cdot 1 + 20$$

The slope of the line is therefore $m = 10 - 20 = -10$ and the required linear function is

$$T = -10h + 20$$

- (b) The graph is sketched in Figure 3. The slope is $m = -10^\circ\text{C}/\text{km}$, and this represents the rate of change of temperature with respect to height.

- (c) At a height of $h = 2.5$ km, the temperature is

$$T = -10(2.5) + 20 = -5^\circ\text{C}$$

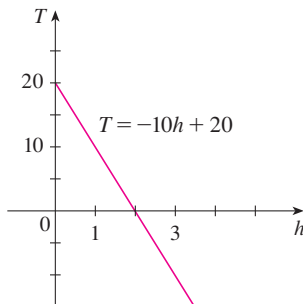


FIGURE 3

If there is no physical law or principle to help us formulate a model, we construct an **empirical model**, which is based entirely on collected data. We seek a curve that “fits” the data in the sense that it captures the basic trend of the data points.

EXAMPLE 2 Table 1 lists the average carbon dioxide level in the atmosphere, measured in parts per million at Mauna Loa Observatory from 1980 to 2002. Use the data in Table 1 to find a model for the carbon dioxide level.

SOLUTION We use the data in Table 1 to make the scatter plot in Figure 4, where t represents time (in years) and C represents the CO_2 level (in parts per million, ppm).

TABLE 1			
Year	CO_2 level (in ppm)	Year	CO_2 level (in ppm)
1980	338.7	1992	356.4
1982	341.1	1994	358.9
1984	344.4	1996	362.6
1986	347.2	1998	366.6
1988	351.5	2000	369.4
1990	354.2	2002	372.9

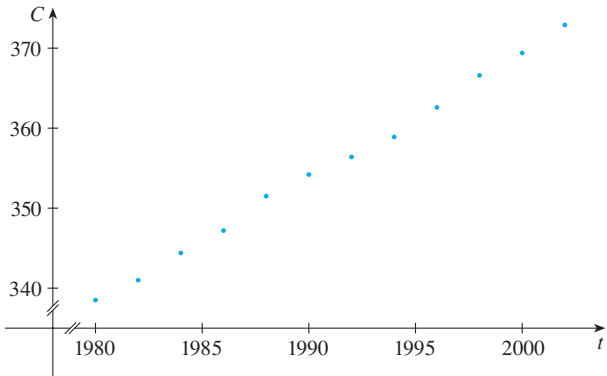


FIGURE 4 Scatter plot for the average CO_2 level

Notice that the data points appear to lie close to a straight line, so it's natural to choose a linear model in this case. But there are many possible lines that approximate these data points, so which one should we use? From the graph, it appears that one possibility is the line that passes through the first and last data points. The slope of this line is

$$\frac{372.9 - 338.7}{2002 - 1980} = \frac{34.2}{22} \approx 1.5545$$

and its equation is

$$C - 338.7 = 1.5545(t - 1980)$$

or

I

$$C = 1.5545t - 2739.21$$

Equation 1 gives one possible linear model for the carbon dioxide level; it is graphed in Figure 5.

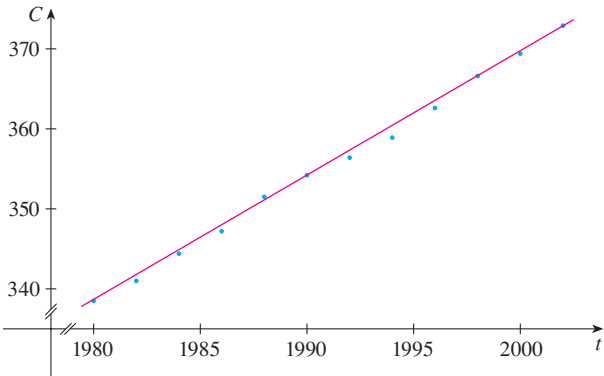


FIGURE 5
Linear model through
first and last data points

Although our model fits the data reasonably well, it gives values higher than most of the actual CO_2 levels. A better linear model is obtained by a procedure from statistics

■ A computer or graphing calculator finds the regression line by the method of **least squares**, which is to minimize the sum of the squares of the vertical distances between the data points and the line. The details are explained in Section 14.7.

called *linear regression*. If we use a graphing calculator, we enter the data from Table 1 into the data editor and choose the linear regression command. (With Maple we use the `fit[leastsquare]` command in the stats package; with Mathematica we use the `Fit` command.) The machine gives the slope and y-intercept of the regression line as

$$m = 1.55192 \quad b = -2734.55$$

So our least squares model for the CO₂ level is

$$\boxed{2} \quad C = 1.55192t - 2734.55$$

In Figure 6 we graph the regression line as well as the data points. Comparing with Figure 5, we see that it gives a better fit than our previous linear model.

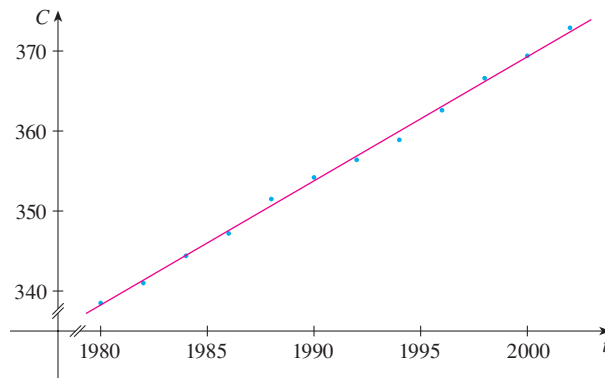


FIGURE 6
The regression line

EXAMPLE 3 Use the linear model given by Equation 2 to estimate the average CO₂ level for 1987 and to predict the level for the year 2010. According to this model, when will the CO₂ level exceed 400 parts per million?

SOLUTION Using Equation 2 with $t = 1987$, we estimate that the average CO₂ level in 1987 was

$$C(1987) = (1.55192)(1987) - 2734.55 \approx 349.12$$

This is an example of *interpolation* because we have estimated a value *between* observed values. (In fact, the Mauna Loa Observatory reported that the average CO₂ level in 1987 was 348.93 ppm, so our estimate is quite accurate.)

With $t = 2010$, we get

$$C(2010) = (1.55192)(2010) - 2734.55 \approx 384.81$$

So we predict that the average CO₂ level in the year 2010 will be 384.8 ppm. This is an example of *extrapolation* because we have predicted a value *outside* the region of observations. Consequently, we are far less certain about the accuracy of our prediction.

Using Equation 2, we see that the CO₂ level exceeds 400 ppm when

$$1.55192t - 2734.55 > 400$$

Solving this inequality, we get

$$t > \frac{3134.55}{1.55192} \approx 2019.79$$

We therefore predict that the CO₂ level will exceed 400 ppm by the year 2019. This prediction is somewhat risky because it involves a time quite remote from our observations.

POLYNOMIALS

A function P is called a **polynomial** if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

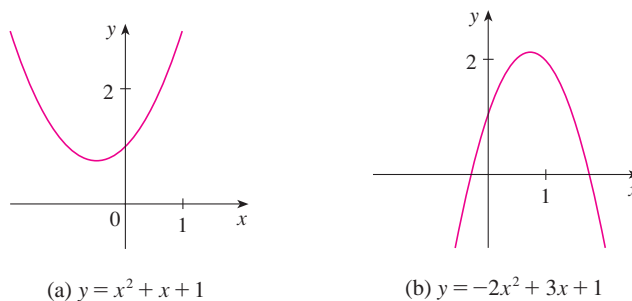
where n is a nonnegative integer and the numbers $a_0, a_1, a_2, \dots, a_n$ are constants called the **coefficients** of the polynomial. The domain of any polynomial is $\mathbb{R} = (-\infty, \infty)$. If the leading coefficient $a_n \neq 0$, then the **degree** of the polynomial is n . For example, the function

$$P(x) = 2x^6 - x^4 + \frac{2}{5}x^3 + \sqrt{2}$$

is a polynomial of degree 6.

A polynomial of degree 1 is of the form $P(x) = mx + b$ and so it is a linear function. A polynomial of degree 2 is of the form $P(x) = ax^2 + bx + c$ and is called a **quadratic function**. Its graph is always a parabola obtained by shifting the parabola $y = ax^2$, as we will see in the next section. The parabola opens upward if $a > 0$ and downward if $a < 0$. (See Figure 7.)

FIGURE 7
The graphs of quadratic functions are parabolas.



A polynomial of degree 3 is of the form

$$P(x) = ax^3 + bx^2 + cx + d \quad (a \neq 0)$$

and is called a **cubic function**. Figure 8 shows the graph of a cubic function in part (a) and graphs of polynomials of degrees 4 and 5 in parts (b) and (c). We will see later why the graphs have these shapes.

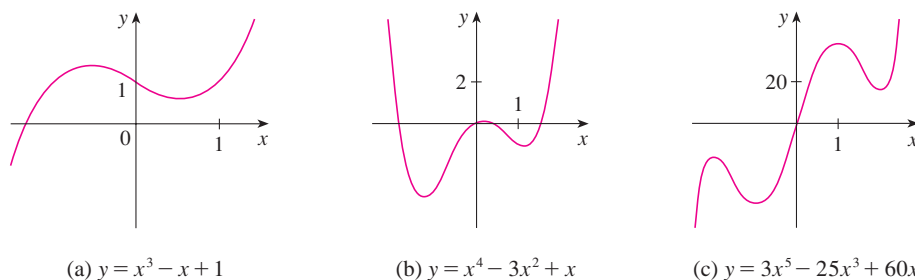


FIGURE 8

Polynomials are commonly used to model various quantities that occur in the natural and social sciences. For instance, in Section 3.7 we will explain why economists often use a polynomial $P(x)$ to represent the cost of producing x units of a commodity. In the following example we use a quadratic function to model the fall of a ball.

TABLE 2

Time (seconds)	Height (meters)
0	450
1	445
2	431
3	408
4	375
5	332
6	279
7	216
8	143
9	61

EXAMPLE 4 A ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground, and its height h above the ground is recorded at 1-second intervals in Table 2. Find a model to fit the data and use the model to predict the time at which the ball hits the ground.

SOLUTION We draw a scatter plot of the data in Figure 9 and observe that a linear model is inappropriate. But it looks as if the data points might lie on a parabola, so we try a quadratic model instead. Using a graphing calculator or computer algebra system (which uses the least squares method), we obtain the following quadratic model:

$$\boxed{3} \quad h = 449.36 + 0.96t - 4.90t^2$$

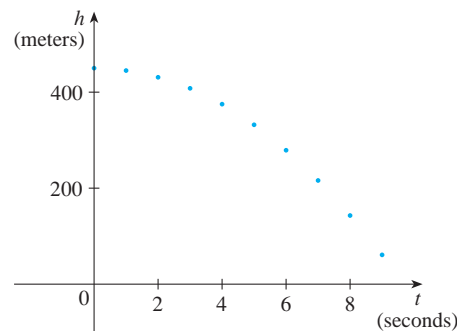


FIGURE 9
Scatter plot for a falling ball

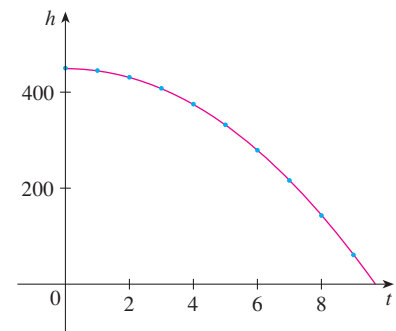


FIGURE 10
Quadratic model for a falling ball

In Figure 10 we plot the graph of Equation 3 together with the data points and see that the quadratic model gives a very good fit.

The ball hits the ground when $h = 0$, so we solve the quadratic equation

$$-4.90t^2 + 0.96t + 449.36 = 0$$

The quadratic formula gives

$$t = \frac{-0.96 \pm \sqrt{(0.96)^2 - 4(-4.90)(449.36)}}{2(-4.90)}$$

The positive root is $t \approx 9.67$, so we predict that the ball will hit the ground after about 9.7 seconds. ■

POWER FUNCTIONS

A function of the form $f(x) = x^a$, where a is a constant, is called a **power function**. We consider several cases.

(i) $a = n$, where n is a positive integer

The graphs of $f(x) = x^n$ for $n = 1, 2, 3, 4$, and 5 are shown in Figure 11. (These are polynomials with only one term.) We already know the shape of the graphs of $y = x$ (a line through the origin with slope 1) and $y = x^2$ [a parabola, see Example 2(b) in Section 1.1].

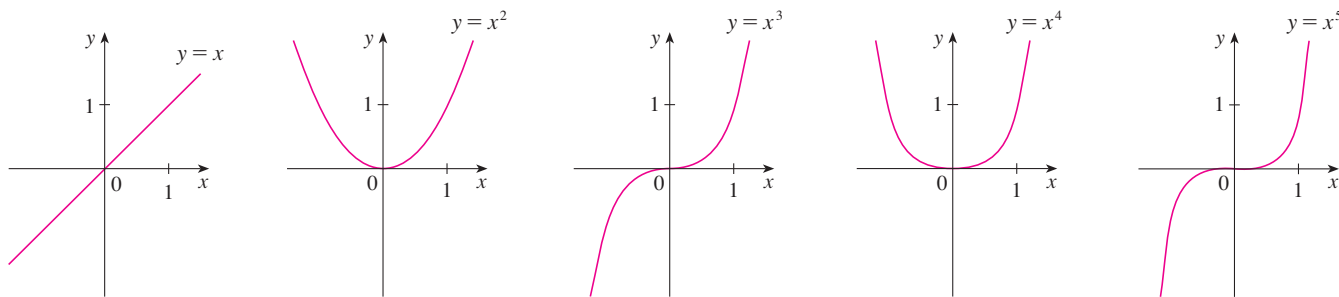


FIGURE 11 Graphs of $f(x) = x^n$ for $n = 1, 2, 3, 4, 5$

The general shape of the graph of $f(x) = x^n$ depends on whether n is even or odd. If n is even, then $f(x) = x^n$ is an even function and its graph is similar to the parabola $y = x^2$. If n is odd, then $f(x) = x^n$ is an odd function and its graph is similar to that of $y = x^3$. Notice from Figure 12, however, that as n increases, the graph of $y = x^n$ becomes flatter near 0 and steeper when $|x| \geq 1$. (If x is small, then x^2 is smaller, x^3 is even smaller, x^4 is smaller still, and so on.)

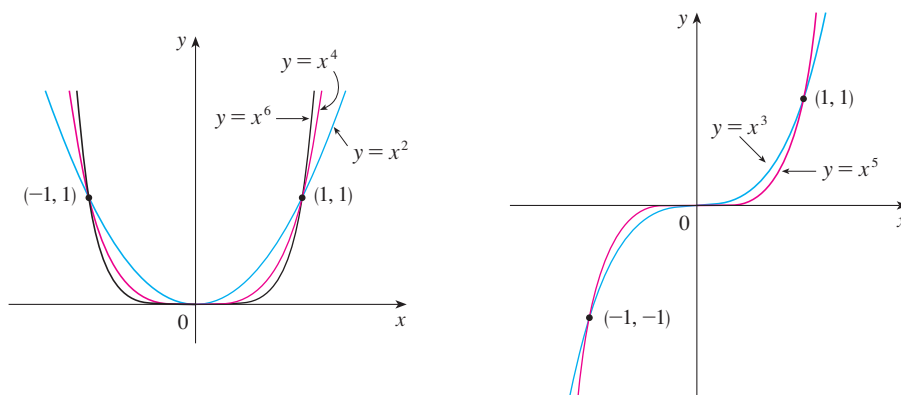


FIGURE 12
Families of power functions

(iii) $a = 1/n$, where n is a positive integer

The function $f(x) = x^{1/n} = \sqrt[n]{x}$ is a **root function**. For $n = 2$ it is the square root function $f(x) = \sqrt{x}$, whose domain is $[0, \infty)$ and whose graph is the upper half of the parabola $x = y^2$. [See Figure 13(a).] For other even values of n , the graph of $y = \sqrt[n]{x}$ is similar to that of $y = \sqrt{x}$. For $n = 3$ we have the cube root function $f(x) = \sqrt[3]{x}$ whose domain is \mathbb{R} (recall that every real number has a cube root) and whose graph is shown in Figure 13(b). The graph of $y = \sqrt[n]{x}$ for n odd ($n > 3$) is similar to that of $y = \sqrt[3]{x}$.

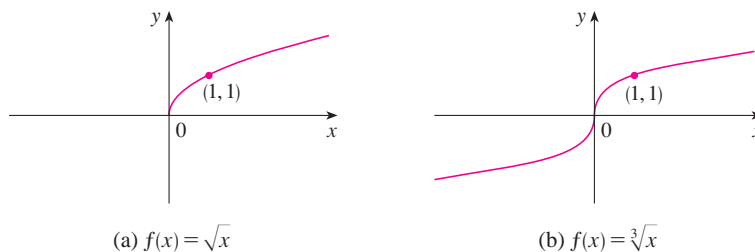


FIGURE 13
Graphs of root functions

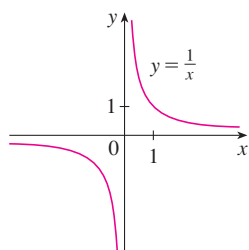


FIGURE 14

The reciprocal function

(iii) $a = -1$

The graph of the **reciprocal function** $f(x) = x^{-1} = 1/x$ is shown in Figure 14. Its graph has the equation $y = 1/x$, or $xy = 1$, and is a hyperbola with the coordinate axes as its asymptotes. This function arises in physics and chemistry in connection with Boyle's Law, which says that, when the temperature is constant, the volume V of a gas is inversely proportional to the pressure P :

$$V = \frac{C}{P}$$

where C is a constant. Thus the graph of V as a function of P (see Figure 15) has the same general shape as the right half of Figure 14.

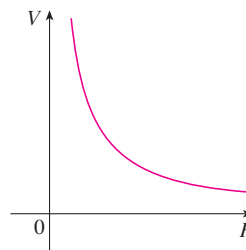


FIGURE 15

Volume as a function of pressure
at constant temperature

Another instance in which a power function is used to model a physical phenomenon is discussed in Exercise 26.

RATIONAL FUNCTIONS

A **rational function** f is a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. The domain consists of all values of x such that $Q(x) \neq 0$. A simple example of a rational function is the function $f(x) = 1/x$, whose domain is $\{x \mid x \neq 0\}$; this is the reciprocal function graphed in Figure 14. The function

$$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$$

is a rational function with domain $\{x \mid x \neq \pm 2\}$. Its graph is shown in Figure 16.

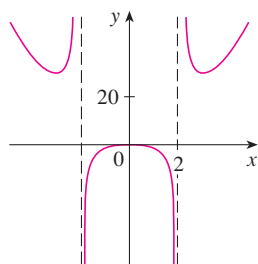


FIGURE 16

$$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$$

ALGEBRAIC FUNCTIONS

A function f is called an **algebraic function** if it can be constructed using algebraic operations (such as addition, subtraction, multiplication, division, and taking roots) starting with polynomials. Any rational function is automatically an algebraic function. Here are two more examples:

$$f(x) = \sqrt{x^2 + 1} \qquad g(x) = \frac{x^4 - 16x^2}{x + \sqrt{x}} + (x - 2)\sqrt[3]{x + 1}$$

When we sketch algebraic functions in Chapter 4, we will see that their graphs can assume a variety of shapes. Figure 17 illustrates some of the possibilities.

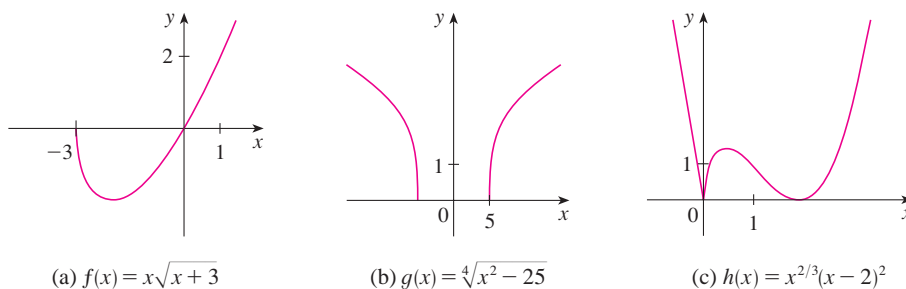


FIGURE 17

An example of an algebraic function occurs in the theory of relativity. The mass of a particle with velocity v is

$$m = f(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the rest mass of the particle and $c = 3.0 \times 10^5$ km/s is the speed of light in a vacuum.

TRIGONOMETRIC FUNCTIONS

■ The Reference Pages are located at the front and back of the book.

Trigonometry and the trigonometric functions are reviewed on Reference Page 2 and also in Appendix D. In calculus the convention is that radian measure is always used (except when otherwise indicated). For example, when we use the function $f(x) = \sin x$, it is understood that $\sin x$ means the sine of the angle whose radian measure is x . Thus the graphs of the sine and cosine functions are as shown in Figure 18.

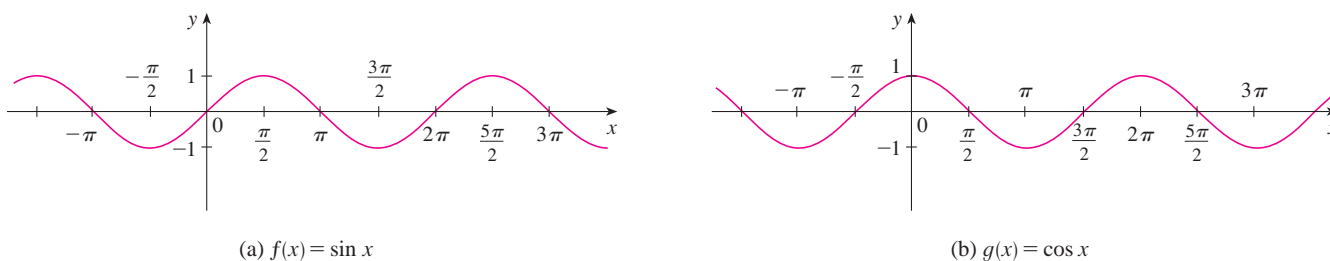


FIGURE 18

Notice that for both the sine and cosine functions the domain is $(-\infty, \infty)$ and the range is the closed interval $[-1, 1]$. Thus, for all values of x , we have

$$-1 \leq \sin x \leq 1 \quad -1 \leq \cos x \leq 1$$

or, in terms of absolute values,

$$|\sin x| \leq 1 \quad |\cos x| \leq 1$$

Also, the zeros of the sine function occur at the integer multiples of π ; that is,

$$\sin x = 0 \quad \text{when} \quad x = n\pi \quad n \text{ an integer}$$

An important property of the sine and cosine functions is that they are periodic functions and have period 2π . This means that, for all values of x ,

$$\sin(x + 2\pi) = \sin x \quad \cos(x + 2\pi) = \cos x$$

The periodic nature of these functions makes them suitable for modeling repetitive phenomena such as tides, vibrating springs, and sound waves. For instance, in Example 4 in Section 1.3 we will see that a reasonable model for the number of hours of daylight in Philadelphia t days after January 1 is given by the function

$$L(t) = 12 + 2.8 \sin \left[\frac{2\pi}{365}(t - 80) \right]$$

The tangent function is related to the sine and cosine functions by the equation

$$\tan x = \frac{\sin x}{\cos x}$$

and its graph is shown in Figure 19. It is undefined whenever $\cos x = 0$, that is, when $x = \pm\pi/2, \pm3\pi/2, \dots$. Its range is $(-\infty, \infty)$. Notice that the tangent function has period π :

$$\tan(x + \pi) = \tan x \quad \text{for all } x$$

The remaining three trigonometric functions (cosecant, secant, and cotangent) are the reciprocals of the sine, cosine, and tangent functions. Their graphs are shown in Appendix D.

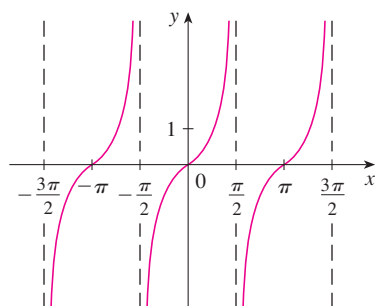


FIGURE 19
 $y = \tan x$

EXPONENTIAL FUNCTIONS

The **exponential functions** are the functions of the form $f(x) = a^x$, where the base a is a positive constant. The graphs of $y = 2^x$ and $y = (0.5)^x$ are shown in Figure 20. In both cases the domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.

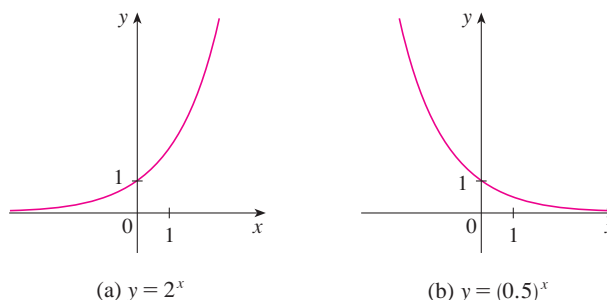


FIGURE 20

Exponential functions will be studied in detail in Section 1.5, and we will see that they are useful for modeling many natural phenomena, such as population growth (if $a > 1$) and radioactive decay (if $a < 1$).

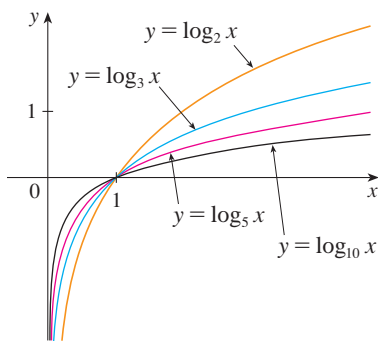


FIGURE 21

LOGARITHMIC FUNCTIONS

The **logarithmic functions** $f(x) = \log_a x$, where the base a is a positive constant, are the inverse functions of the exponential functions. They will be studied in Section 1.6. Figure 21 shows the graphs of four logarithmic functions with various bases. In each case the domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the function increases slowly when $x > 1$.

TRANSCENDENTAL FUNCTIONS

These are functions that are not algebraic. The set of transcendental functions includes the trigonometric, inverse trigonometric, exponential, and logarithmic functions, but it also includes a vast number of other functions that have never been named. In Chapter 11 we will study transcendental functions that are defined as sums of infinite series.

EXAMPLE 5 Classify the following functions as one of the types of functions that we have discussed.

(a) $f(x) = 5^x$

(b) $g(x) = x^5$

(c) $h(x) = \frac{1+x}{1-\sqrt{x}}$

(d) $u(t) = 1 - t + 5t^4$

SOLUTION

(a) $f(x) = 5^x$ is an exponential function. (The x is the exponent.)

(b) $g(x) = x^5$ is a power function. (The x is the base.) We could also consider it to be a polynomial of degree 5.

(c) $h(x) = \frac{1+x}{1-\sqrt{x}}$ is an algebraic function.

(d) $u(t) = 1 - t + 5t^4$ is a polynomial of degree 4. ■

1.2 EXERCISES

1–2 Classify each function as a power function, root function, polynomial (state its degree), rational function, algebraic function, trigonometric function, exponential function, or logarithmic function.

1. (a) $f(x) = \sqrt[5]{x}$

(b) $g(x) = \sqrt{1-x^2}$

(c) $h(x) = x^9 + x^4$

(d) $r(x) = \frac{x^2 + 1}{x^3 + x}$

(e) $s(x) = \tan 2x$

(f) $t(x) = \log_{10} x$

2. (a) $y = \frac{x-6}{x+6}$

(b) $y = x + \frac{x^2}{\sqrt{x-1}}$

(c) $y = 10^x$

(d) $y = x^{10}$

(e) $y = 2t^6 + t^4 - \pi$

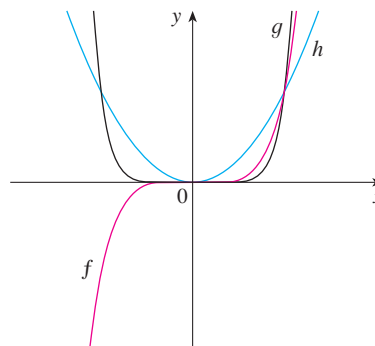
(f) $y = \cos \theta + \sin \theta$

3–4 Match each equation with its graph. Explain your choices. (Don't use a computer or graphing calculator.)

3. (a) $y = x^2$

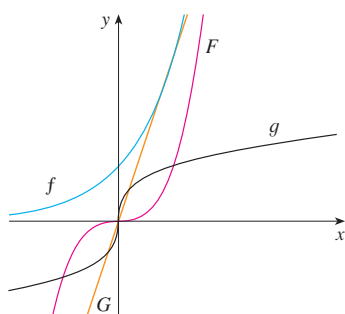
(b) $y = x^5$

(c) $y = x^8$

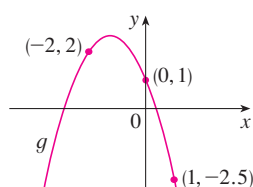
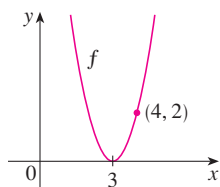


4. (a) $y = 3x$
(c) $y = x^3$

- (b) $y = 3^x$
(d) $y = \sqrt[3]{x}$



5. (a) Find an equation for the family of linear functions with slope 2 and sketch several members of the family.
(b) Find an equation for the family of linear functions such that $f(2) = 1$ and sketch several members of the family.
(c) Which function belongs to both families?
6. What do all members of the family of linear functions $f(x) = 1 + m(x + 3)$ have in common? Sketch several members of the family.
7. What do all members of the family of linear functions $f(x) = c - x$ have in common? Sketch several members of the family.
8. Find expressions for the quadratic functions whose graphs are shown.

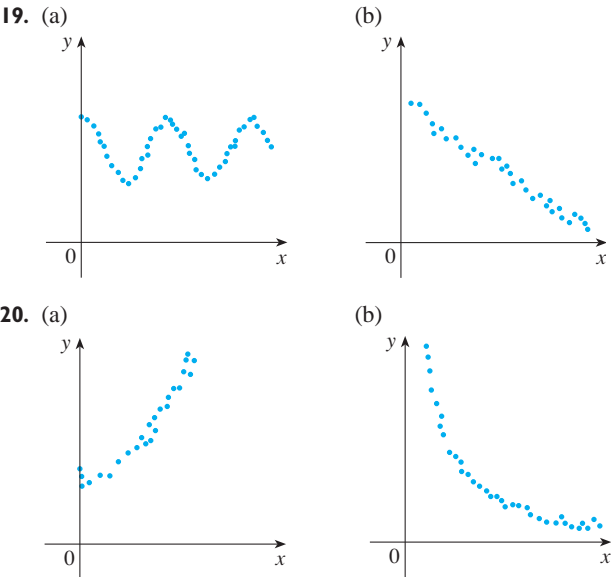


9. Find an expression for a cubic function f if $f(1) = 6$ and $f(-1) = f(0) = f(2) = 0$.
10. Recent studies indicate that the average surface temperature of the earth has been rising steadily. Some scientists have modeled the temperature by the linear function $T = 0.02t + 8.50$, where T is temperature in $^{\circ}\text{C}$ and t represents years since 1900.
(a) What do the slope and T -intercept represent?
(b) Use the equation to predict the average global surface temperature in 2100.
11. If the recommended adult dosage for a drug is D (in mg), then to determine the appropriate dosage c for a child of age a , pharmacists use the equation $c = 0.0417D(a + 1)$. Suppose the dosage for an adult is 200 mg.
(a) Find the slope of the graph of c . What does it represent?
(b) What is the dosage for a newborn?

12. The manager of a weekend flea market knows from past experience that if he charges x dollars for a rental space at the market, then the number y of spaces he can rent is given by the equation $y = 200 - 4x$.
(a) Sketch a graph of this linear function. (Remember that the rental charge per space and the number of spaces rented can't be negative quantities.)
(b) What do the slope, the y -intercept, and the x -intercept of the graph represent?
13. The relationship between the Fahrenheit (F) and Celsius (C) temperature scales is given by the linear function $F = \frac{9}{5}C + 32$.
(a) Sketch a graph of this function.
(b) What is the slope of the graph and what does it represent? What is the F -intercept and what does it represent?
14. Jason leaves Detroit at 2:00 PM and drives at a constant speed west along I-96. He passes Ann Arbor, 40 mi from Detroit, at 2:50 PM.
(a) Express the distance traveled in terms of the time elapsed.
(b) Draw the graph of the equation in part (a).
(c) What is the slope of this line? What does it represent?
15. Biologists have noticed that the chirping rate of crickets of a certain species is related to temperature, and the relationship appears to be very nearly linear. A cricket produces 113 chirps per minute at 70°F and 173 chirps per minute at 80°F .
(a) Find a linear equation that models the temperature T as a function of the number of chirps per minute N .
(b) What is the slope of the graph? What does it represent?
(c) If the crickets are chirping at 150 chirps per minute, estimate the temperature.
16. The manager of a furniture factory finds that it costs \$2200 to manufacture 100 chairs in one day and \$4800 to produce 300 chairs in one day.
(a) Express the cost as a function of the number of chairs produced, assuming that it is linear. Then sketch the graph.
(b) What is the slope of the graph and what does it represent?
(c) What is the y -intercept of the graph and what does it represent?
17. At the surface of the ocean, the water pressure is the same as the air pressure above the water, 15 lb/in^2 . Below the surface, the water pressure increases by 4.34 lb/in^2 for every 10 ft of descent.
(a) Express the water pressure as a function of the depth below the ocean surface.
(b) At what depth is the pressure 100 lb/in^2 ?

18. The monthly cost of driving a car depends on the number of miles driven. Lynn found that in May it cost her \$380 to drive 480 mi and in June it cost her \$460 to drive 800 mi.
- (a) Express the monthly cost C as a function of the distance driven d , assuming that a linear relationship gives a suitable model.
 - (b) Use part (a) to predict the cost of driving 1500 miles per month.
 - (c) Draw the graph of the linear function. What does the slope represent?
 - (d) What does the y -intercept represent?
 - (e) Why does a linear function give a suitable model in this situation?

19–20 For each scatter plot, decide what type of function you might choose as a model for the data. Explain your choices.



21. The table shows (lifetime) peptic ulcer rates (per 100 population) for various family incomes as reported by the National Health Interview Survey.

Income	Ulcer rate (per 100 population)
\$4,000	14.1
\$6,000	13.0
\$8,000	13.4
\$12,000	12.5
\$16,000	12.0
\$20,000	12.4
\$30,000	10.5
\$45,000	9.4
\$60,000	8.2

- (a) Make a scatter plot of these data and decide whether a linear model is appropriate.

- (b) Find and graph a linear model using the first and last data points.
- (c) Find and graph the least squares regression line.
- (d) Use the linear model in part (c) to estimate the ulcer rate for an income of \$25,000.
- (e) According to the model, how likely is someone with an income of \$80,000 to suffer from peptic ulcers?
- (f) Do you think it would be reasonable to apply the model to someone with an income of \$200,000?

22. Biologists have observed that the chirping rate of crickets of a certain species appears to be related to temperature. The table shows the chirping rates for various temperatures.

Temperature (°F)	Chirping rate (chirps/min)	Temperature (°F)	Chirping rate (chirps/min)
50	20	75	140
55	46	80	173
60	79	85	198
65	91	90	211
70	113		

- (a) Make a scatter plot of the data.
- (b) Find and graph the regression line.
- (c) Use the linear model in part (b) to estimate the chirping rate at 100°F.

23. The table gives the winning heights for the Olympic pole vault competitions in the 20th century.

Year	Height (ft)	Year	Height (ft)
1900	10.83	1956	14.96
1904	11.48	1960	15.42
1908	12.17	1964	16.73
1912	12.96	1968	17.71
1920	13.42	1972	18.04
1924	12.96	1976	18.04
1928	13.77	1980	18.96
1932	14.15	1984	18.85
1936	14.27	1988	19.77
1948	14.10	1992	19.02
1952	14.92	1996	19.42

- (a) Make a scatter plot and decide whether a linear model is appropriate.
- (b) Find and graph the regression line.
- (c) Use the linear model to predict the height of the winning pole vault at the 2000 Olympics and compare with the actual winning height of 19.36 feet.
- (d) Is it reasonable to use the model to predict the winning height at the 2100 Olympics?

24. A study by the US Office of Science and Technology in 1972 estimated the cost (in 1972 dollars) to reduce automobile emissions by certain percentages:

Reduction in emissions (%)	Cost per car (in \$)	Reduction in emissions (%)	Cost per car (in \$)
50	45	75	90
55	55	80	100
60	62	85	200
65	70	90	375
70	80	95	600

Find a model that captures the “diminishing returns” trend of these data.

25. Use the data in the table to model the population of the world in the 20th century by a cubic function. Then use your model to estimate the population in the year 1925.

Year	Population (millions)	Year	Population (millions)
1900	1650	1960	3040
1910	1750	1970	3710
1920	1860	1980	4450
1930	2070	1990	5280
1940	2300	2000	6080
1950	2560		

26. The table shows the mean (average) distances d of the planets from the sun (taking the unit of measurement to be the distance from the earth to the sun) and their periods T (time of revolution in years).

Planet	d	T
Mercury	0.387	0.241
Venus	0.723	0.615
Earth	1.000	1.000
Mars	1.523	1.881
Jupiter	5.203	11.861
Saturn	9.541	29.457
Uranus	19.190	84.008
Neptune	30.086	164.784

- (a) Fit a power model to the data.
(b) Kepler’s Third Law of Planetary Motion states that

“The square of the period of revolution of a planet is proportional to the cube of its mean distance from the sun.”

Does your model corroborate Kepler’s Third Law?

1.3

NEW FUNCTIONS FROM OLD FUNCTIONS

In this section we start with the basic functions we discussed in Section 1.2 and obtain new functions by shifting, stretching, and reflecting their graphs. We also show how to combine pairs of functions by the standard arithmetic operations and by composition.

TRANSFORMATIONS OF FUNCTIONS

By applying certain transformations to the graph of a given function we can obtain the graphs of certain related functions. This will give us the ability to sketch the graphs of many functions quickly by hand. It will also enable us to write equations for given graphs. Let’s first consider **translations**. If c is a positive number, then the graph of $y = f(x) + c$ is just the graph of $y = f(x)$ shifted upward a distance of c units (because each y -coordinate is increased by the same number c). Likewise, if $g(x) = f(x - c)$, where $c > 0$, then the value of g at x is the same as the value of f at $x - c$ (c units to the left of x). Therefore, the graph of $y = f(x - c)$ is just the graph of $y = f(x)$ shifted c units to the right (see Figure 1).

VERTICAL AND HORIZONTAL SHIFTS Suppose $c > 0$. To obtain the graph of

- $y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward
- $y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward
- $y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right
- $y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left

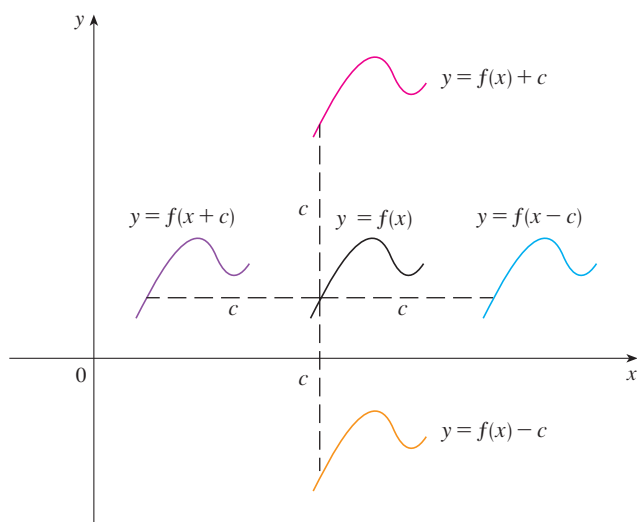


FIGURE 1
Translating the graph of f

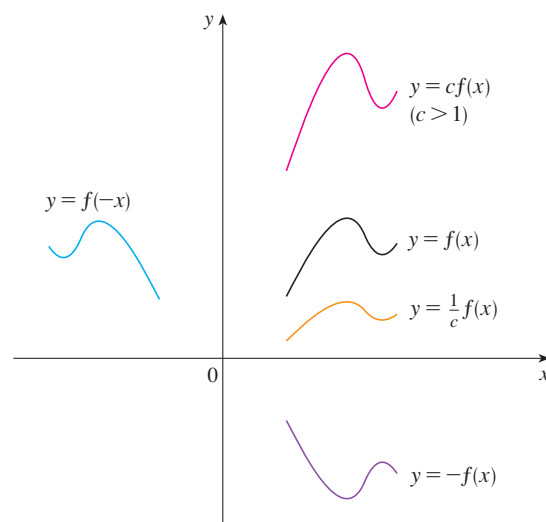


FIGURE 2
Stretching and reflecting the graph of f

Now let's consider the **stretching** and **reflecting** transformations. If $c > 1$, then the graph of $y = cf(x)$ is the graph of $y = f(x)$ stretched by a factor of c in the vertical direction (because each y -coordinate is multiplied by the same number c). The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected about the x -axis because the point (x, y) is replaced by the point $(x, -y)$. (See Figure 2 and the following chart, where the results of other stretching, compressing, and reflecting transformations are also given.)

VERTICAL AND HORIZONTAL STRETCHING AND REFLECTING Suppose $c > 1$. To obtain the graph of

- $y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c
- $y = (1/c)f(x)$, compress the graph of $y = f(x)$ vertically by a factor of c
- $y = f(cx)$, compress the graph of $y = f(x)$ horizontally by a factor of c
- $y = f(x/c)$, stretch the graph of $y = f(x)$ horizontally by a factor of c
- $y = -f(x)$, reflect the graph of $y = f(x)$ about the x -axis
- $y = f(-x)$, reflect the graph of $y = f(x)$ about the y -axis

Figure 3 illustrates these stretching transformations when applied to the cosine function with $c = 2$. For instance, in order to get the graph of $y = 2 \cos x$ we multiply the y -coordinate of each point on the graph of $y = \cos x$ by 2. This means that the graph of $y = \cos x$ gets stretched vertically by a factor of 2.

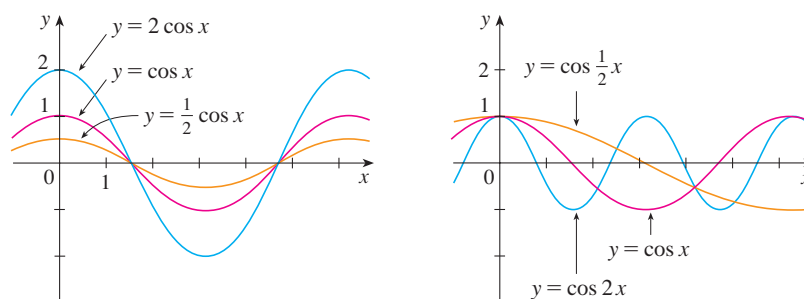


FIGURE 3

EXAMPLE 1 Given the graph of $y = \sqrt{x}$, use transformations to graph $y = \sqrt{x} - 2$, $y = \sqrt{x} - 2$, $y = -\sqrt{x}$, $y = 2\sqrt{x}$, and $y = \sqrt{-x}$.

SOLUTION The graph of the square root function $y = \sqrt{x}$, obtained from Figure 13(a) in Section 1.2, is shown in Figure 4(a). In the other parts of the figure we sketch $y = \sqrt{x} - 2$ by shifting 2 units downward, $y = \sqrt{x - 2}$ by shifting 2 units to the right, $y = -\sqrt{x}$ by reflecting about the x -axis, $y = 2\sqrt{x}$ by stretching vertically by a factor of 2, and $y = \sqrt{-x}$ by reflecting about the y -axis.

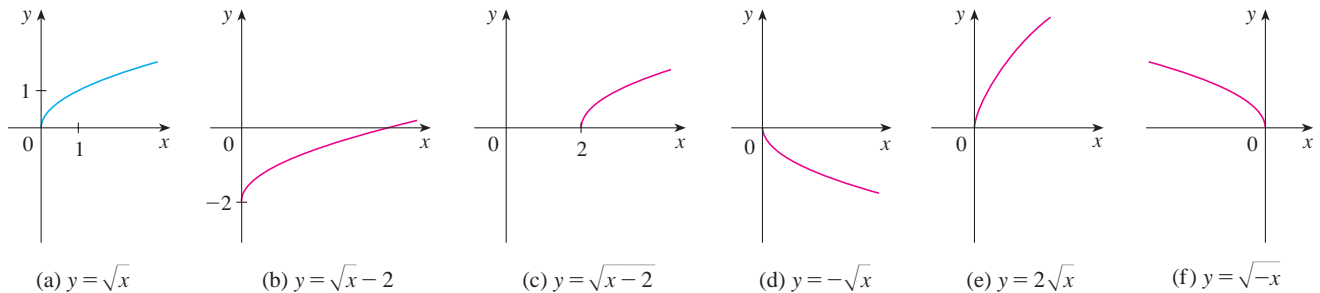


FIGURE 4

EXAMPLE 2 Sketch the graph of the function $f(x) = x^2 + 6x + 10$.

SOLUTION Completing the square, we write the equation of the graph as

$$y = x^2 + 6x + 10 = (x + 3)^2 + 1$$

This means we obtain the desired graph by starting with the parabola $y = x^2$ and shifting 3 units to the left and then 1 unit upward (see Figure 5).

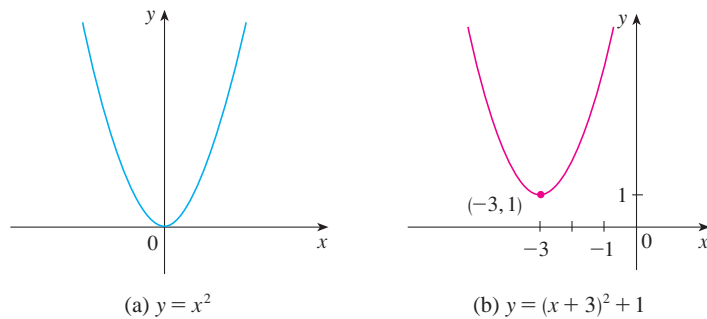


FIGURE 5

EXAMPLE 3 Sketch the graphs of the following functions.

(a) $y = \sin 2x$

(b) $y = 1 - \sin x$

SOLUTION

(a) We obtain the graph of $y = \sin 2x$ from that of $y = \sin x$ by compressing horizontally by a factor of 2 (see Figures 6 and 7). Thus, whereas the period of $y = \sin x$ is 2π , the period of $y = \sin 2x$ is $2\pi/2 = \pi$.

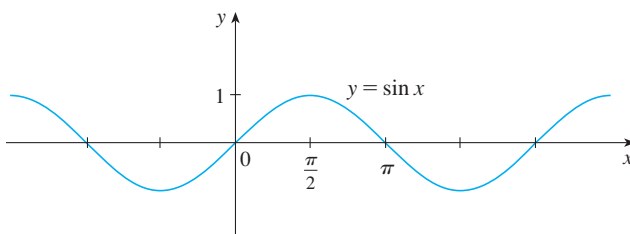


FIGURE 6

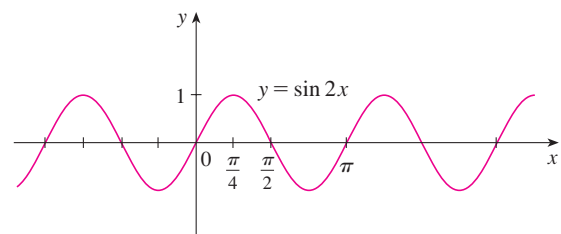


FIGURE 7

(b) To obtain the graph of $y = 1 - \sin x$, we again start with $y = \sin x$. We reflect about the x -axis to get the graph of $y = -\sin x$ and then we shift 1 unit upward to get $y = 1 - \sin x$. (See Figure 8.)

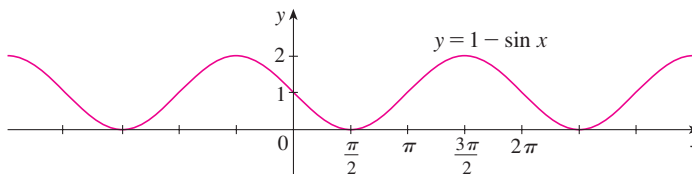


FIGURE 8

EXAMPLE 4 Figure 9 shows graphs of the number of hours of daylight as functions of the time of the year at several latitudes. Given that Philadelphia is located at approximately 40°N latitude, find a function that models the length of daylight at Philadelphia.

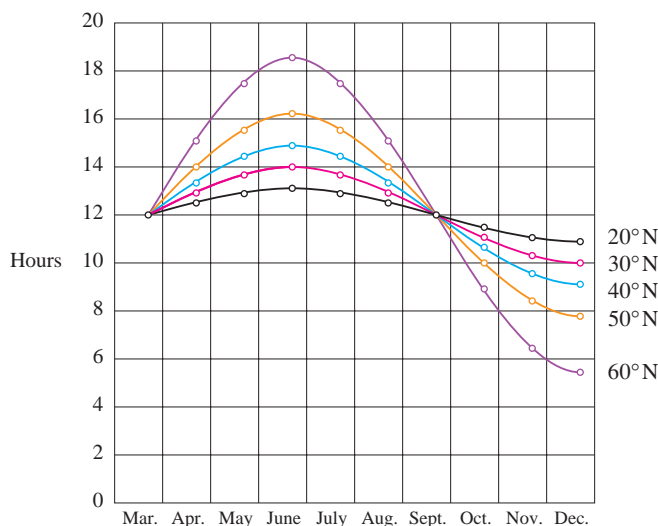


FIGURE 9

Graph of the length of daylight
from March 21 through December 21
at various latitudes

Lucia C. Harrison, *Daylight, Twilight, Darkness and Time*
(New York: Silver, Burdett, 1935) page 40.

SOLUTION Notice that each curve resembles a shifted and stretched sine function. By looking at the blue curve we see that, at the latitude of Philadelphia, daylight lasts about 14.8 hours on June 21 and 9.2 hours on December 21, so the amplitude of the curve (the factor by which we have to stretch the sine curve vertically) is $\frac{1}{2}(14.8 - 9.2) = 2.8$.

By what factor do we need to stretch the sine curve horizontally if we measure the time t in days? Because there are about 365 days in a year, the period of our model should be 365. But the period of $y = \sin t$ is 2π , so the horizontal stretching factor is $c = 2\pi/365$.

We also notice that the curve begins its cycle on March 21, the 80th day of the year, so we have to shift the curve 80 units to the right. In addition, we shift it 12 units upward. Therefore we model the length of daylight in Philadelphia on the t th day of the year by the function

$$L(t) = 12 + 2.8 \sin \left[\frac{2\pi}{365}(t - 80) \right]$$

Another transformation of some interest is taking the *absolute value* of a function. If $y = |f(x)|$, then according to the definition of absolute value, $y = f(x)$ when $f(x) \geq 0$ and $y = -f(x)$ when $f(x) < 0$. This tells us how to get the graph of $y = |f(x)|$ from the graph of $y = f(x)$: The part of the graph that lies above the x -axis remains the same; the part that lies below the x -axis is reflected about the x -axis.

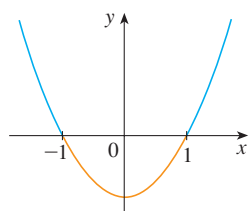
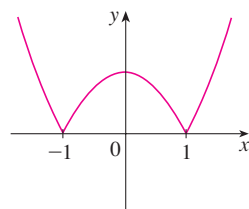
(a) $y = x^2 - 1$ (b) $y = |x^2 - 1|$

FIGURE 10

EXAMPLE 5 Sketch the graph of the function $y = |x^2 - 1|$.

SOLUTION We first graph the parabola $y = x^2 - 1$ in Figure 10(a) by shifting the parabola $y = x^2$ downward 1 unit. We see that the graph lies below the x -axis when $-1 < x < 1$, so we reflect that part of the graph about the x -axis to obtain the graph of $y = |x^2 - 1|$ in Figure 10(b).

COMBINATIONS OF FUNCTIONS

Two functions f and g can be combined to form new functions $f + g$, $f - g$, fg , and f/g in a manner similar to the way we add, subtract, multiply, and divide real numbers. The sum and difference functions are defined by

$$(f + g)(x) = f(x) + g(x) \quad (f - g)(x) = f(x) - g(x)$$

If the domain of f is A and the domain of g is B , then the domain of $f + g$ is the intersection $A \cap B$ because both $f(x)$ and $g(x)$ have to be defined. For example, the domain of $f(x) = \sqrt{x}$ is $A = [0, \infty)$ and the domain of $g(x) = \sqrt{2 - x}$ is $B = (-\infty, 2]$, so the domain of $(f + g)(x) = \sqrt{x} + \sqrt{2 - x}$ is $A \cap B = [0, 2]$.

Similarly, the product and quotient functions are defined by

$$(fg)(x) = f(x)g(x) \quad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

The domain of fg is $A \cap B$, but we can't divide by 0 and so the domain of f/g is $\{x \in A \cap B \mid g(x) \neq 0\}$. For instance, if $f(x) = x^2$ and $g(x) = x - 1$, then the domain of the rational function $(f/g)(x) = x^2/(x - 1)$ is $\{x \mid x \neq 1\}$, or $(-\infty, 1) \cup (1, \infty)$.

There is another way of combining two functions to obtain a new function. For example, suppose that $y = f(u) = \sqrt{u}$ and $u = g(x) = x^2 + 1$. Since y is a function of u and u is, in turn, a function of x , it follows that y is ultimately a function of x . We compute this by substitution:

$$y = f(u) = f(g(x)) = f(x^2 + 1) = \sqrt{x^2 + 1}$$

The procedure is called *composition* because the new function is *composed* of the two given functions f and g .

In general, given any two functions f and g , we start with a number x in the domain of g and find its image $g(x)$. If this number $g(x)$ is in the domain of f , then we can calculate the value of $f(g(x))$. The result is a new function $h(x) = f(g(x))$ obtained by substituting g into f . It is called the *composition* (or *composite*) of f and g and is denoted by $f \circ g$ (“ f circle g ”).

DEFINITION Given two functions f and g , the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$

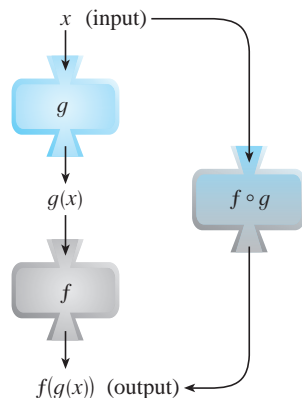


FIGURE 11

The $f \circ g$ machine is composed of the g machine (first) and then the f machine.


The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f . In other words, $(f \circ g)(x)$ is defined whenever both $g(x)$ and $f(g(x))$ are defined. Figure 11 shows how to picture $f \circ g$ in terms of machines.

EXAMPLE 6 If $f(x) = x^2$ and $g(x) = x - 3$, find the composite functions $f \circ g$ and $g \circ f$.

SOLUTION We have

$$(f \circ g)(x) = f(g(x)) = f(x - 3) = (x - 3)^2$$

$$(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 - 3$$

 **NOTE** You can see from Example 6 that, in general, $f \circ g \neq g \circ f$. Remember, the notation $f \circ g$ means that the function g is applied first and then f is applied second. In Example 6, $f \circ g$ is the function that *first* subtracts 3 and *then* squares; $g \circ f$ is the function that *first* squares and *then* subtracts 3.

EXAMPLE 7 If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2 - x}$, find each function and its domain.

- (a) $f \circ g$ (b) $g \circ f$ (c) $f \circ f$ (d) $g \circ g$

SOLUTION

$$(a) \quad (f \circ g)(x) = f(g(x)) = f(\sqrt{2 - x}) = \sqrt{\sqrt{2 - x}} = \sqrt[4]{2 - x}$$

The domain of $f \circ g$ is $\{x \mid 2 - x \geq 0\} = \{x \mid x \leq 2\} = (-\infty, 2]$.

$$(b) \quad (g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = \sqrt{2 - \sqrt{x}}$$

For \sqrt{x} to be defined we must have $x \geq 0$. For $\sqrt{2 - \sqrt{x}}$ to be defined we must have $2 - \sqrt{x} \geq 0$, that is, $\sqrt{x} \leq 2$, or $x \leq 4$. Thus we have $0 \leq x \leq 4$, so the domain of $g \circ f$ is the closed interval $[0, 4]$.

$$(c) \quad (f \circ f)(x) = f(f(x)) = f(\sqrt{x}) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$$

The domain of $f \circ f$ is $[0, \infty)$.

$$(d) \quad (g \circ g)(x) = g(g(x)) = g(\sqrt{2 - x}) = \sqrt{2 - \sqrt{2 - x}}$$

This expression is defined when both $2 - x \geq 0$ and $2 - \sqrt{2 - x} \geq 0$. The first inequality means $x \leq 2$, and the second is equivalent to $\sqrt{2 - x} \leq 2$, or $2 - x \leq 4$, or $x \geq -2$. Thus $-2 \leq x \leq 2$, so the domain of $g \circ g$ is the closed interval $[-2, 2]$.

It is possible to take the composition of three or more functions. For instance, the composite function $f \circ g \circ h$ is found by first applying h , then g , and then f as follows:

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

EXAMPLE 8 Find $f \circ g \circ h$ if $f(x) = x/(x + 1)$, $g(x) = x^{10}$, and $h(x) = x + 3$.

SOLUTION

$$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x + 3))$$

$$= f((x + 3)^{10}) = \frac{(x + 3)^{10}}{(x + 3)^{10} + 1}$$

So far we have used composition to build complicated functions from simpler ones. But in calculus it is often useful to be able to *decompose* a complicated function into simpler ones, as in the following example.

If $0 \leq a \leq b$, then $a^2 \leq b^2$.

EXAMPLE 9 Given $F(x) = \cos^2(x + 9)$, find functions f , g , and h such that $F = f \circ g \circ h$.

SOLUTION Since $F(x) = [\cos(x + 9)]^2$, the formula for F says: First add 9, then take the cosine of the result, and finally square. So we let

$$h(x) = x + 9 \quad g(x) = \cos x \quad f(x) = x^2$$

Then

$$\begin{aligned}(f \circ g \circ h)(x) &= f(g(h(x))) = f(g(x + 9)) = f(\cos(x + 9)) \\ &= [\cos(x + 9)]^2 = F(x)\end{aligned}$$

1.3 EXERCISES

- 1.** Suppose the graph of f is given. Write equations for the graphs that are obtained from the graph of f as follows.

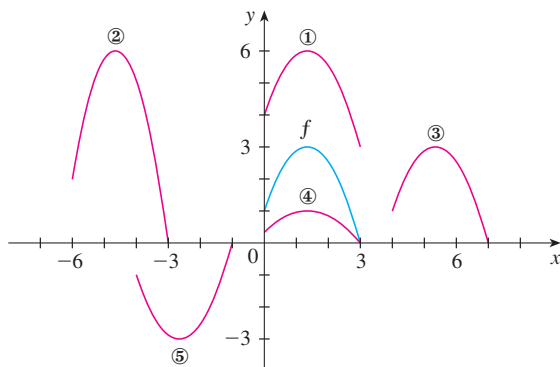
- Shift 3 units upward.
- Shift 3 units downward.
- Shift 3 units to the right.
- Shift 3 units to the left.
- Reflect about the x -axis.
- Reflect about the y -axis.
- Stretch vertically by a factor of 3.
- Shrink vertically by a factor of 3.

- 2.** Explain how each graph is obtained from the graph of $y = f(x)$.

- $y = 5f(x)$
- $y = f(x - 5)$
- $y = -f(x)$
- $y = -5f(x)$
- $y = f(5x)$
- $y = 5f(x) - 3$

- 3.** The graph of $y = f(x)$ is given. Match each equation with its graph and give reasons for your choices.

- $y = f(x - 4)$
- $y = f(x) + 3$
- $y = \frac{1}{3}f(x)$
- $y = -f(x + 4)$
- $y = 2f(x + 6)$

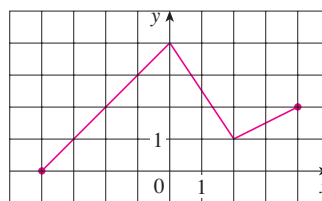


- 4.** The graph of f is given. Draw the graphs of the following functions.

- $y = f(x + 4)$
- $y = f(x) + 4$

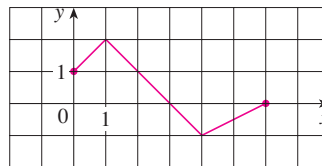
(c) $y = 2f(x)$

(d) $y = -\frac{1}{2}f(x) + 3$

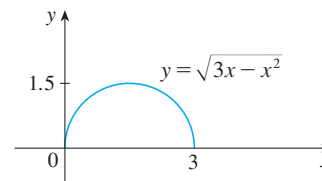


- 5.** The graph of f is given. Use it to graph the following functions.

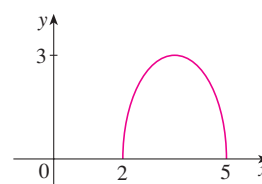
- $y = f(2x)$
- $y = f(\frac{1}{2}x)$
- $y = f(-x)$
- $y = -f(-x)$



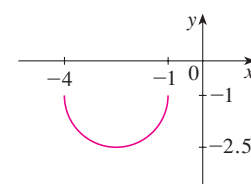
- 6–7** The graph of $y = \sqrt{3x - x^2}$ is given. Use transformations to create a function whose graph is as shown.



6.



7.



8. (a) How is the graph of $y = 2 \sin x$ related to the graph of $y = \sin x$? Use your answer and Figure 6 to sketch the graph of $y = 2 \sin x$.
 (b) How is the graph of $y = 1 + \sqrt{x}$ related to the graph of $y = \sqrt{x}$? Use your answer and Figure 4(a) to sketch the graph of $y = 1 + \sqrt{x}$.

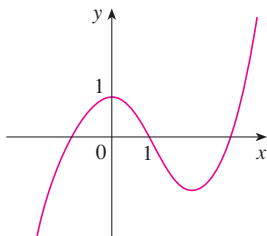
9–24 Graph the function by hand, not by plotting points, but by starting with the graph of one of the standard functions given in Section 1.2, and then applying the appropriate transformations.

9. $y = -x^3$ 10. $y = 1 - x^2$
 11. $y = (x + 1)^2$ 12. $y = x^2 - 4x + 3$
 13. $y = 1 + 2 \cos x$ 14. $y = 4 \sin 3x$
 15. $y = \sin(x/2)$ 16. $y = \frac{1}{x - 4}$
 17. $y = \sqrt{x + 3}$ 18. $y = (x + 2)^4 + 3$
 19. $y = \frac{1}{2}(x^2 + 8x)$ 20. $y = 1 + \sqrt[3]{x - 1}$
 21. $y = \frac{2}{x + 1}$ 22. $y = \frac{1}{4} \tan\left(x - \frac{\pi}{4}\right)$
 23. $y = |\sin x|$ 24. $y = |x^2 - 2x|$

25. The city of New Orleans is located at latitude 30°N . Use Figure 9 to find a function that models the number of hours of daylight at New Orleans as a function of the time of year. To check the accuracy of your model, use the fact that on March 31 the sun rises at 5:51 AM and sets at 6:18 PM in New Orleans.

26. A variable star is one whose brightness alternately increases and decreases. For the most visible variable star, Delta Cephei, the time between periods of maximum brightness is 5.4 days, the average brightness (or magnitude) of the star is 4.0, and its brightness varies by ± 0.35 magnitude. Find a function that models the brightness of Delta Cephei as a function of time.

27. (a) How is the graph of $y = f(|x|)$ related to the graph of f ?
 (b) Sketch the graph of $y = \sin |x|$.
 (c) Sketch the graph of $y = \sqrt{|x|}$.
 28. Use the given graph of f to sketch the graph of $y = 1/f(x)$. Which features of f are the most important in sketching $y = 1/f(x)$? Explain how they are used.



29–30 Find $f + g$, $f - g$, fg , and f/g and state their domains.

29. $f(x) = x^3 + 2x^2$, $g(x) = 3x^2 - 1$

30. $f(x) = \sqrt{3 - x}$, $g(x) = \sqrt{x^2 - 1}$

31–36 Find the functions (a) $f \circ g$, (b) $g \circ f$, (c) $f \circ f$, and (d) $g \circ g$ and their domains.

31. $f(x) = x^2 - 1$, $g(x) = 2x + 1$

32. $f(x) = x - 2$, $g(x) = x^2 + 3x + 4$

33. $f(x) = 1 - 3x$, $g(x) = \cos x$

34. $f(x) = \sqrt{x}$, $g(x) = \sqrt[3]{1 - x}$

35. $f(x) = x + \frac{1}{x}$, $g(x) = \frac{x + 1}{x + 2}$

36. $f(x) = \frac{x}{1 + x}$, $g(x) = \sin 2x$

37–40 Find $f \circ g \circ h$.

37. $f(x) = x + 1$, $g(x) = 2x$, $h(x) = x - 1$

38. $f(x) = 2x - 1$, $g(x) = x^2$, $h(x) = 1 - x$

39. $f(x) = \sqrt{x - 3}$, $g(x) = x^2$, $h(x) = x^3 + 2$

40. $f(x) = \tan x$, $g(x) = \frac{x}{x - 1}$, $h(x) = \sqrt[3]{x}$

41–46 Express the function in the form $f \circ g$.

41. $F(x) = (x^2 + 1)^{10}$

42. $F(x) = \sin(\sqrt{x})$

43. $F(x) = \frac{\sqrt[3]{x}}{1 + \sqrt[3]{x}}$

44. $G(x) = \sqrt[3]{\frac{x}{1 + x}}$

45. $u(t) = \sqrt{\cos t}$

46. $u(t) = \frac{\tan t}{1 + \tan t}$

47–49 Express the function in the form $f \circ g \circ h$.

47. $H(x) = 1 - 3x^2$

48. $H(x) = \sqrt[8]{2 + |x|}$

49. $H(x) = \sec^4(\sqrt{x})$

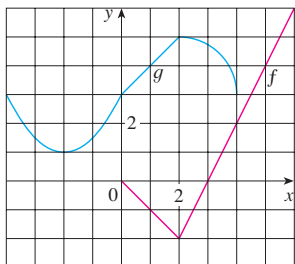
50. Use the table to evaluate each expression.

- (a) $f(g(1))$ (b) $g(f(1))$ (c) $f(f(1))$
 (d) $g(g(1))$ (e) $(g \circ f)(3)$ (f) $(f \circ g)(6)$

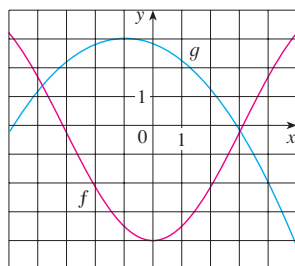
x	1	2	3	4	5	6
$f(x)$	3	1	4	2	2	5
$g(x)$	6	3	2	1	2	3

51. Use the given graphs of f and g to evaluate each expression, or explain why it is undefined.

- (a) $f(g(2))$ (b) $g(f(0))$ (c) $(f \circ g)(0)$
 (d) $(g \circ f)(6)$ (e) $(g \circ g)(-2)$ (f) $(f \circ f)(4)$



52. Use the given graphs of f and g to estimate the value of $f(g(x))$ for $x = -5, -4, -3, \dots, 5$. Use these estimates to sketch a rough graph of $f \circ g$.



53. A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 60 cm/s.
 (a) Express the radius r of this circle as a function of the time t (in seconds).
 (b) If A is the area of this circle as a function of the radius, find $A \circ r$ and interpret it.
54. A spherical balloon is being inflated and the radius of the balloon is increasing at a rate of 2 cm/s.
 (a) Express the radius r of the balloon as a function of the time t (in seconds).
 (b) If V is the volume of the balloon as a function of the radius, find $V \circ r$ and interpret it.
55. A ship is moving at a speed of 30 km/h parallel to a straight shoreline. The ship is 6 km from shore and it passes a lighthouse at noon.
 (a) Express the distance s between the lighthouse and the ship as a function of d , the distance the ship has traveled since noon; that is, find f so that $s = f(d)$.
 (b) Express d as a function of t , the time elapsed since noon; that is, find g so that $d = g(t)$.
 (c) Find $f \circ g$. What does this function represent?
56. An airplane is flying at a speed of 350 mi/h at an altitude of one mile and passes directly over a radar station at time $t = 0$.
 (a) Express the horizontal distance d (in miles) that the plane has flown as a function of t .
 (b) Express the distance s between the plane and the radar station as a function of d .
 (c) Use composition to express s as a function of t .

57. The **Heaviside function** H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

It is used in the study of electric circuits to represent the sudden surge of electric current, or voltage, when a switch is instantaneously turned on.

- (a) Sketch the graph of the Heaviside function.
 (b) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 0$ and 120 volts are applied instantaneously to the circuit. Write a formula for $V(t)$ in terms of $H(t)$.
 (c) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 5$ seconds and 240 volts are applied instantaneously to the circuit. Write a formula for $V(t)$ in terms of $H(t)$. (Note that starting at $t = 5$ corresponds to a translation.)
58. The Heaviside function defined in Exercise 57 can also be used to define the **ramp function** $y = ctH(t)$, which represents a gradual increase in voltage or current in a circuit.
 (a) Sketch the graph of the ramp function $y = tH(t)$.
 (b) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 0$ and the voltage is gradually increased to 120 volts over a 60-second time interval. Write a formula for $V(t)$ in terms of $H(t)$ for $t \leq 60$.
 (c) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 7$ seconds and the voltage is gradually increased to 100 volts over a period of 25 seconds. Write a formula for $V(t)$ in terms of $H(t)$ for $t \leq 32$.
59. Let f and g be linear functions with equations $f(x) = m_1x + b_1$ and $g(x) = m_2x + b_2$. Is $f \circ g$ also a linear function? If so, what is the slope of its graph?
60. If you invest x dollars at 4% interest compounded annually, then the amount $A(x)$ of the investment after one year is $A(x) = 1.04x$. Find $A \circ A$, $A \circ A \circ A$, and $A \circ A \circ A \circ A$. What do these compositions represent? Find a formula for the composition of n copies of A .
61. (a) If $g(x) = 2x + 1$ and $h(x) = 4x^2 + 4x + 7$, find a function f such that $f \circ g = h$. (Think about what operations you would have to perform on the formula for g to end up with the formula for h .)
 (b) If $f(x) = 3x + 5$ and $h(x) = 3x^2 + 3x + 2$, find a function g such that $f \circ g = h$.
62. If $f(x) = x + 4$ and $h(x) = 4x - 1$, find a function g such that $g \circ f = h$.
63. (a) Suppose f and g are even functions. What can you say about $f + g$ and fg ?
 (b) What if f and g are both odd?
64. Suppose f is even and g is odd. What can you say about fg ?
65. Suppose g is an even function and let $h = f \circ g$. Is h always an even function?
66. Suppose g is an odd function and let $h = f \circ g$. Is h always an odd function? What if f is odd? What if f is even?